7.4 Translations and Vectors

What *you should learn*

GOAL 1 Identify and use translations in the plane.

GOAL 2 Use vectors in **real-life** situations, such as navigating a sailboat in **Example 6**.

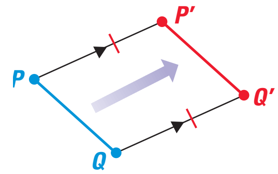
Why *you should learn it*

▼ You can use translations and vectors to describe the path of an aircraft, such as the hot-air balloon in **Exs. 53–55**.

REAL LIFE

GOAL 1 USING PROPERTIES OF TRANSLATIONS

A **translation** is a transformation that maps every two points P and Q in the plane to points P' and Q', so that the following properties are true:



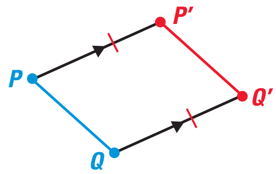
1. $PP' = QQ'$
2. $\overline{PP'} \parallel \overline{QQ'}$, or $\overline{PP'}$ and $\overline{QQ'}$ are collinear.

THEOREM

THEOREM 7.4 ***Translation Theorem***

A translation is an isometry.

Theorem 7.4 can be proven as follows.

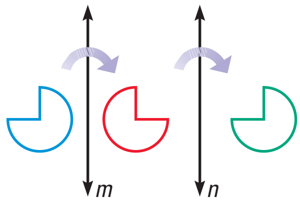


GIVEN ▶ $PP' = QQ'$, $\overline{PP'} \parallel \overline{QQ'}$

PROVE ▶ $PQ = P'Q'$

Paragraph Proof The quadrilateral $PP'Q'Q$ has a pair of opposite sides that are congruent and parallel, which implies $PP'Q'Q$ is a parallelogram. From this you can conclude $PQ = P'Q'$. (Exercise 43 asks for a coordinate proof of Theorem 7.4, which covers the case where $\overline{PQ}$ and $\overline{P'Q'}$ are collinear.)

You can find the image of a translation by gliding a figure in the plane. Another way to find the image of a translation is to complete one reflection after another in two parallel lines, as shown. The properties of this type of translation are stated below.

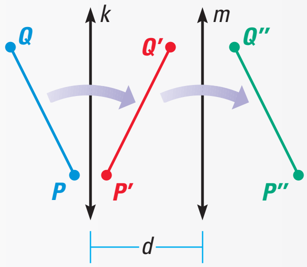


THEOREM

THEOREM 7.5

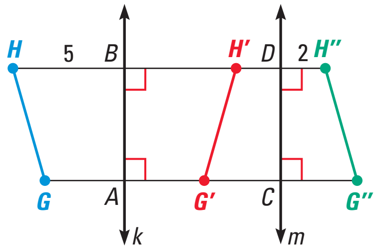
If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is a translation. If P'' is the image of P, then the following is true:

1. $\overleftrightarrow{PP''}$ is perpendicular to k and m.
2. $PP'' = 2d$, where d is the distance between k and m.



EXAMPLE 1 Using Theorem 7.5

In the diagram, a reflection in line k maps $\overline{GH}$ to $\overline{G'H'}$, a reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$, $k \parallel m$, $HB = 5$, and $DH'' = 2$.



a. Name some congruent segments.

b. Does $AC = BD$? Explain.

c. What is the length of $\overline{GG''}$?

Student Help

Study Tip

In Lesson 7.2, you learned that the line of reflection is the perpendicular bisector of the segment connecting a point and its image. In Example 1, you can use this property to conclude that figure *ABDC* is a rectangle.

SOLUTION

a. Here are some sets of congruent segments: $\overline{GH}$, $\overline{G'H'}$, and $\overline{G''H''}$; $\overline{HB}$ and $\overline{H'B}$; $\overline{H'D}$ and $\overline{H''D}$.

b. Yes, $AC = BD$ because $\overline{AC}$ and $\overline{BD}$ are opposite sides of a rectangle.

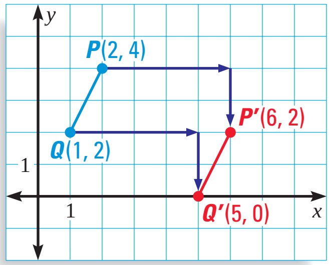
c. Because $GG'' = HH''$, the length of $\overline{GG''}$ is $5 + 5 + 2 + 2$, or 14 units.

....

Translations in a coordinate plane can be described by the following coordinate notation:

$$(x, y) \rightarrow (x + a, y + b)$$

where a and b are constants. Each point shifts a units horizontally and b units vertically. For instance, in the coordinate plane at the right, the translation $(x, y) \rightarrow (x + 4, y - 2)$ shifts each point 4 units to the right and 2 units down.



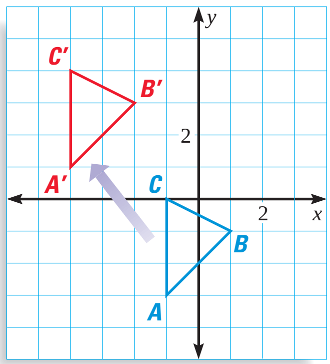
EXAMPLE 2 Translations in a Coordinate Plane

Sketch a triangle with vertices $A(-1, -3)$, $B(1, -1)$, and $C(-1, 0)$. Then sketch the image of the triangle after the translation $(x, y) \rightarrow (x - 3, y + 4)$.

SOLUTION

Plot the points as shown. Shift each point 3 units to the left and 4 units up to find the translated vertices. The coordinates of the vertices of the preimage and image are listed below.

$\triangle ABC$	$\triangle A'B'C'$
$A(-1, -3)$	$A'(-4, 1)$
$B(1, -1)$	$B'(-2, 3)$
$C(-1, 0)$	$C'(-4, 4)$

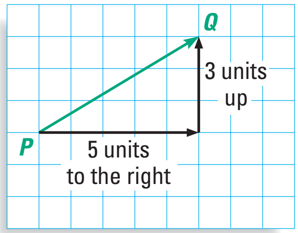


Notice that each x-coordinate of the image is 3 units less than the x-coordinate of the preimage and each y-coordinate of the image is 4 units more than the y-coordinate of the preimage.

> **STUDENT HELP**
>
> **Study Tip**
> When writing a vector in component form, use the correct brackets. The brackets used to write the component form of a vector are different than the parentheses used to write an ordered pair.

GOAL 2 TRANSLATIONS USING VECTORS

Another way to describe a translation is by using a vector. A **vector** is a quantity that has both direction and *magnitude*, or size, and is represented by an arrow drawn between two points.



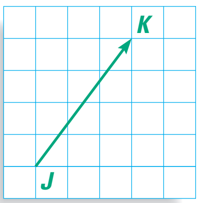
The diagram shows a vector. The **initial point**, or starting point, of the vector is P and the **terminal point**, or ending point, is Q. The vector is named $\overrightarrow{PQ}$, which is read as "vector PQ." The *horizontal component* of $\overrightarrow{PQ}$ is 5 and the *vertical component* is 3.

The **component form** of a vector combines the horizontal and vertical components. So, the component form of $\overrightarrow{PQ}$ is $\langle 5, 3 \rangle$.

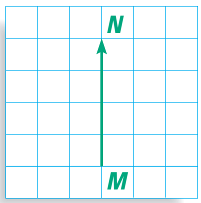
EXAMPLE 3 *Identifying Vector Components*

In the diagram, name each vector and write its component form.

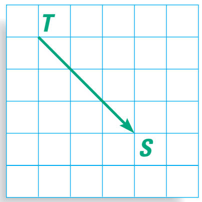
a.



b.



c.



SOLUTION

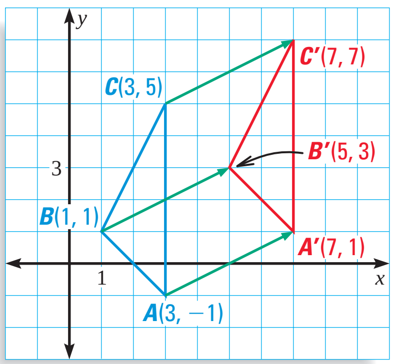
a. The vector is $\overrightarrow{JK}$. To move from the initial point J to the terminal point K, you move 3 units to the right and 4 units up. So, the component form is $\langle 3, 4 \rangle$.

b. The vector is $\overrightarrow{MN} = \langle 0, 4 \rangle$.

c. The vector is $\overrightarrow{TS} = \langle 3, -3 \rangle$.

EXAMPLE 4 *Translation Using Vectors*

The component form of $\overrightarrow{GH}$ is $\langle 4, 2 \rangle$. Use $\overrightarrow{GH}$ to translate the triangle whose vertices are $A(3, -1)$, $B(1, 1)$, and $C(3, 5)$.



SOLUTION

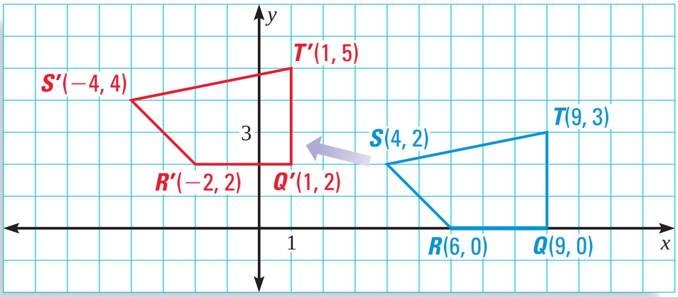
First graph $\triangle ABC$. The component form of $\overrightarrow{GH}$ is $\langle 4, 2 \rangle$, so the image vertices should all be 4 units to the right and 2 units up from the preimage vertices. Label the image vertices as $A'(7, 1)$, $B'(5, 3)$, and $C'(7, 7)$. Then, using a straightedge, draw $\triangle A'B'C'$. Notice that the vectors drawn from preimage to image vertices are parallel.

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

EXAMPLE 5 *Finding Vectors*

In the diagram, $QRST$ maps onto $Q'R'S'T'$ by a translation. Write the component form of the vector that can be used to describe the translation.



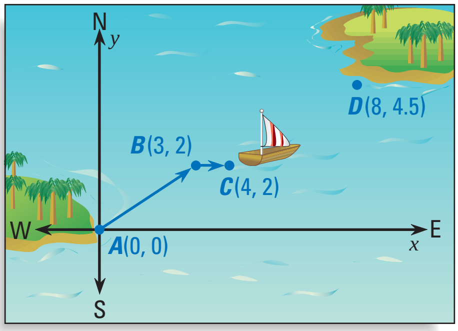
SOLUTION

Choose any vertex and its image, say R and R'. To move from R to R', you move 8 units to the left and 2 units up. The component form of the vector is $\langle -8, 2\rangle$.

✓**CHECK** To check the solution, you can start any where on the preimage and move 8 units to the left and 2 units up. You should end on the corresponding point of the image.

EXAMPLE 6 *Using Vectors*

REAL LIFE **NAVIGATION** A boat travels a straight path between two islands, A and D. When the boat is 3 miles east and 2 miles north of its starting point it encounters a storm at point B. The storm pushes the boat off course to point C, as shown.



a. Write the component forms of the two vectors shown in the diagram.

b. The final destination is 8 miles east and 4.5 miles north of the starting point. Write the component form of the vector that describes the path the boat can follow to arrive at its destination.

SOLUTION

a. The component form of the vector from $A(0, 0)$ to $B(3, 2)$ is

$$\overrightarrow{AB} = \langle 3 - 0, 2 - 0\rangle = \langle 3, 2\rangle.$$

The component form of the vector from $B(3, 2)$ to $C(4, 2)$ is

$$\overrightarrow{BC} = \langle 4 - 3, 2 - 2\rangle = \langle 1, 0\rangle.$$

b. The boat needs to travel from its current position, point C, to the island, point D. To find the component form of the vector from $C(4, 2)$ to $D(8, 4.5)$, subtract the corresponding coordinates:

$$\overrightarrow{CD} = \langle 8 - 4, 4.5 - 2\rangle = \langle 4, 2.5\rangle.$$

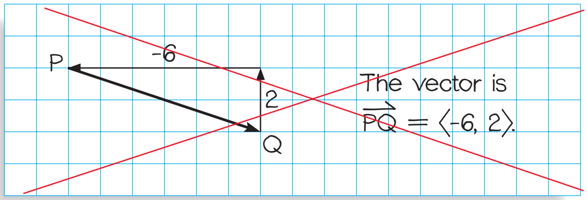
GUIDED PRACTICE

Vocabulary Check ✓

1. A __?__ is a quantity that has both __?__ and magnitude.

Concept Check ✓

2. ERROR ANALYSIS Describe Jerome's error.



Skill Check ✓

Use coordinate notation to describe the translation.

3. 6 units to the right and 2 units down

4. 3 units up and 4 units to the right

5. 7 units to the left and 1 unit up

6. 8 units down and 5 units to the left

Complete the statement using the description of the translation. In the description, points (0, 2) and (8, 5) are two vertices of a pentagon.

7. If (0, 2) maps onto (0, 0), then (8, 5) maps onto (__?__, __?__).

8. If (0, 2) maps onto (__?__, __?__), then (8, 5) maps onto (3, 7).

9. If (0, 2) maps onto $(-3, -5)$, then (8, 5) maps onto (__?__, __?__).

10. If (0, 2) maps onto (__?__, __?__), then (8, 5) maps onto (0, 0).

Draw three vectors that can be described by the given component form.

11. $\langle 3, 5 \rangle$

12. $\langle 0, 4 \rangle$

13. $\langle -6, 0 \rangle$

14. $\langle -5, -1 \rangle$

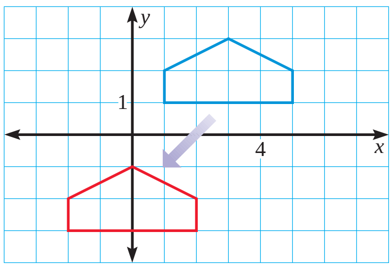
PRACTICE AND APPLICATIONS

STUDENT HELP

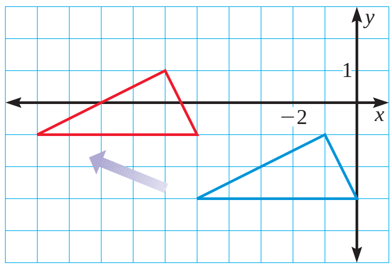
Extra Practice to help you master skills is on p. 816.

DESCRIBING TRANSLATIONS **Describe the translation using (a) coordinate notation and (b) a vector in component form.**

15.



16.



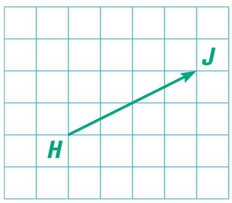
STUDENT HELP

HOMEWORK HELP

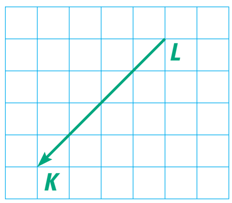
Example 1: Exs. 20–24
Example 2: Exs. 15, 16, 25–34
Example 3: Exs. 15–19
Example 4: Exs. 39–42
Example 5: Exs. 44–47
Example 6: Exs. 53–55

IDENTIFYING VECTORS **Name the vector and write its component form.**

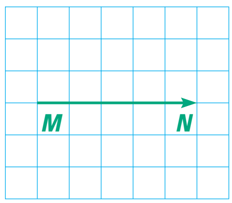
17.



18.



19.



USING THEOREM 7.5 **In the diagram, $k \parallel m$, $\triangle ABC$ is reflected in line k, and $\triangle A'B'C'$ is reflected in line m.**

20. A translation maps $\triangle ABC$ onto which triangle?

21. Which lines are perpendicular to $\overleftrightarrow{AA''}$?

22. Name two segments parallel to $\overline{BB''}$.

23. If the distance between k and m is 1.4 inches, what is the length of $\overline{CC''}$?

24. Is the distance from B' to m the same as the distance from B'' to m? Explain.

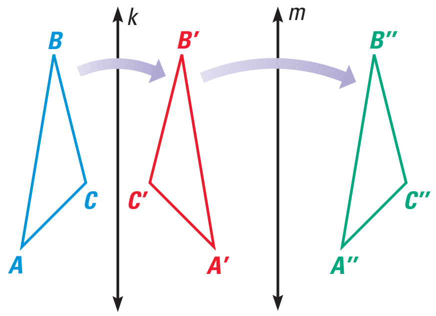


IMAGE AND PREIMAGE **Consider the translation that is defined by the coordinate notation $(x, y) \to (x + 12, y - 7)$.**

25. What is the image of $(5, 3)$?

26. What is the image of $(-1, -2)$?

27. What is the preimage of $(-2, 1)$?

28. What is the preimage of $(0, -6)$?

29. What is the image of $(0.5, 2.5)$?

30. What is the preimage of $(-5.5, -5.5)$?

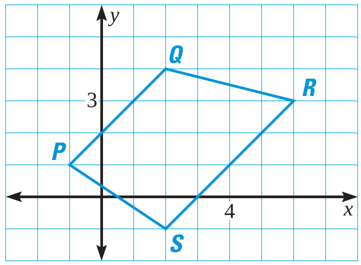
DRAWING AN IMAGE **Copy figure *PQRS* and draw its image after the translation.**

31. $(x, y) \to (x + 1, y - 4)$

32. $(x, y) \to (x - 6, y + 7)$

33. $(x, y) \to (x + 5, y - 2)$

34. $(x, y) \to (x - 1, y - 3)$



STUDENT HELP

INTERNET

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 35–38.

LOGICAL REASONING **Use a straightedge and graph paper to help determine whether the statement is true.**

35. If line p is a translation of a different line q, then p is parallel to q.

36. It is possible for a translation to map a line p onto a perpendicular line q.

37. If a translation maps $\triangle ABC$ onto $\triangle DEF$ and a translation maps $\triangle DEF$ onto $\triangle GHK$, then a translation maps $\triangle ABC$ onto $\triangle GHK$.

38. If a translation maps $\triangle ABC$ onto $\triangle DEF$, then $AD = BE = CF$.

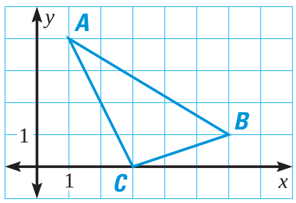
TRANSLATING A TRIANGLE **In Exercises 39–42, use a straightedge and graph paper to translate $\triangle ABC$ by the given vector.**

39. $\langle 2, 4 \rangle$

40. $\langle 3, -2 \rangle$

41. $\langle -1, -5 \rangle$

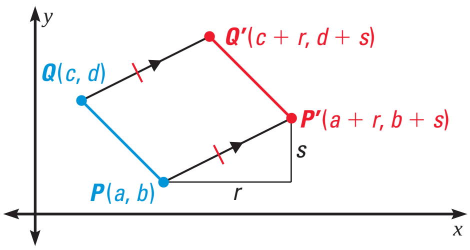
42. $\langle -4, 1 \rangle$



43. **PROOF** Use coordinate geometry and the Distance Formula to write a paragraph proof of Theorem 7.4.

GIVEN ▶ $PP' = QQ'$ and $\overline{PP'} \parallel \overline{QQ'}$

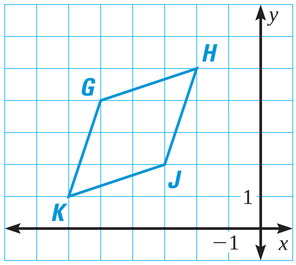
PROVE ▶ $PQ = P'Q'$



VECTORS The vertices of the image of *GHJK* after a translation are given. Choose the vector that describes the translation.

A. $\overrightarrow{PQ} = \langle 1, -3 \rangle$ **B.** $\overrightarrow{PQ} = \langle 0, 1 \rangle$

C. $\overrightarrow{PQ} = \langle -1, -3 \rangle$ **D.** $\overrightarrow{PQ} = \langle 6, -1 \rangle$



44. $G'(-6, 1), H'(-3, 2), J'(-4, -1), K'(-7, -2)$

45. $G'(1, 3), H'(4, 4), J'(3, 1), K'(0, 0)$

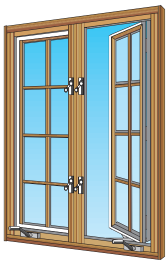
46. $G'(-4, 1), H'(-1, 2), J'(-2, -1), K'(-5, -2)$

47. $G'(-5, 5), H'(-2, 6), J'(-3, 3), K'(-6, 2)$

WINDOW FRAMES In Exercises 48–50, decide whether "opening the window" is a translation of the moving part.

48. Double hung

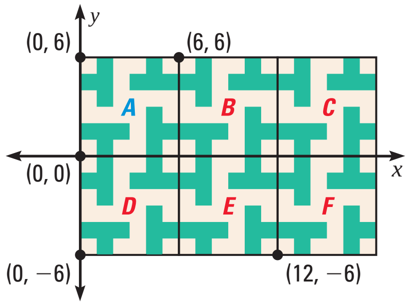
49. Casement



50. Sliding

51. DATA COLLECTION Look through some newspapers and magazines to find patterns containing translations.

52. COMPUTER-AIDED DESIGN Mosaic floors can be designed on a computer. An example is shown at the right. On the computer, the design in square A is copied to cover an entire floor. The translation $(x, y) \rightarrow (x + 6, y)$ maps square A onto square B. Use coordinate notation to describe the translations that map square A onto squares C, D, E, and F.



FOCUS ON APPLICATIONS

REAL LIFE

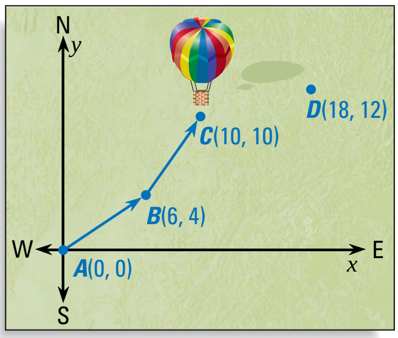
HOT-AIR BALLOONS
Bertrand Piccard and Brian Jones journeyed around the world in their hot-air balloon in 19 days.

NAVIGATION A hot-air balloon is flying from town *A* to town *D*. After the balloon leaves town *A* and travels 6 miles east and 4 miles north, it runs into some heavy winds at point *B*. The balloon is blown off course as shown in the diagram.

53. Write the component forms of the two vectors in the diagram.

54. Write the component form of the vector that describes the path the balloon can take to arrive in town D.

55. Suppose the balloon was not blown off course. Write the component form of the vector that describes this journey from town A to town D.



Test Preparation

QUANTITATIVE COMPARISON In Exercises 56–59, choose the statement that is true about the given quantities.

Ⓐ The quantity in column A is greater.

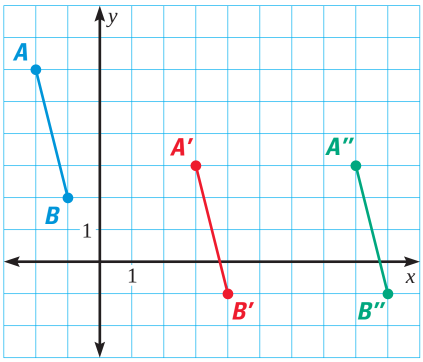
Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

The translation $(x, y) \to (x + 5, y - 3)$ maps $\overline{AB}$ to $\overline{A'B'}$, and the translation $(x, y) \to (x + 5, y)$ maps $\overline{A'B'}$ to $\overline{A''B''}$.

	Column A	Column B
56.	AB	$A'B'$
57.	AB	AA'
58.	BB'	$A'A''$
59.	$A'B''$	$A''B'$



★ Challenge

USING ALGEBRA A translation of $\overline{AB}$ is described by $\overrightarrow{PQ}$. Find the value of each variable.

60. $\overrightarrow{PQ} = \langle 4, 1\rangle$

$A(-1, w)$, $A'(2x + 1, 4)$

$B(8y - 1, 1)$, $B'(3, 3z)$

61. $\overrightarrow{PQ} = \langle 3, -6\rangle$

$A(r - 1, 8)$, $A'(3, s + 1)$

$B(2t - 2, u)$, $B'(5, -2u)$

EXTRA CHALLENGE
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MIXED REVIEW

FINDING SLOPE Find the slope of the line that passes through the given points. (Review 3.6)

62. $A(0, -2)$, $B(-7, -8)$ **63.** $C(2, 3)$, $D(-1, 18)$ **64.** $E(-10, 1)$, $F(-1, 1)$

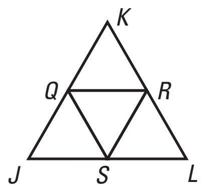
65. $G(-2, 12)$, $H(-1, 6)$ **66.** $J(-6, 0)$, $K(0, 10)$ **67.** $M(-3, -3)$, $N(9, 6)$

COMPLETING THE STATEMENT In $\triangle JKL$, points Q, R, and S are midpoints of the sides. (Review 5.4)

68. If $JK = 12$, then $SR = \underline{\ ?\ }$.

69. If $QR = 6$, then $JL = \underline{\ ?\ }$.

70. If $RL = 6$, then $QS = \underline{\ ?\ }$.



REFLECTIONS IN A COORDINATE PLANE Decide whether the statement is *true* or *false*. (Review 7.2 for 7.5)

71. If $N(3, 4)$ is reflected in the line $y = -1$, then N' is $(3, -6)$.

72. If $M(-5, 3)$ is reflected in the line $x = -2$, then M' is $(3, 1)$.

73. If $W(4, 3)$ is reflected in the line $y = 2$, then W' is $(1, 4)$.