

7.3

Rotations

What you should learn

GOAL 1 Identify rotations in a plane.

GOAL 2 Use rotational symmetry in real-life situations, such as the logo designs in Example 5.

Why you should learn it

Rotations and rotational symmetry can be used to create a design, as in the wheel hubs below and in Exs. 36–38.



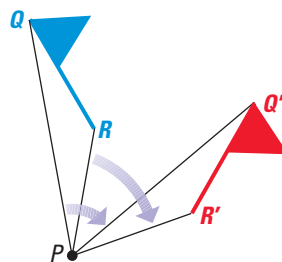
GOAL 1 USING ROTATIONS

A **rotation** is a transformation in which a figure is turned about a fixed point. The fixed point is the **center of rotation**. Rays drawn from the center of rotation to a point and its image form an angle called the **angle of rotation**.

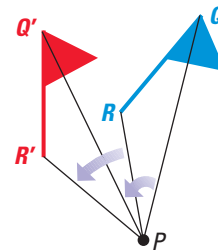
A rotation about a point P through x degrees (x°) is a transformation that maps every point Q in the plane to a point Q' , so that the following properties are true:

1. If Q is not point P , then $QP = Q'P$ and $m\angle QPQ' = x^\circ$.
2. If Q is point P , then $Q = Q'$.

Rotations can be clockwise or counterclockwise, as shown below.



Clockwise rotation of 60°



Counterclockwise rotation of 40°

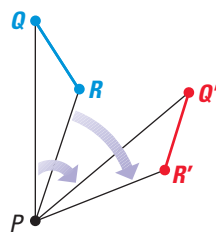
THEOREM

THEOREM 7.2 Rotation Theorem

A rotation is an isometry.

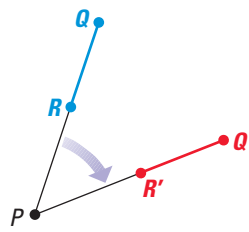
To prove the Rotation Theorem, you need to show that a rotation preserves the length of a segment. Consider a segment \overline{QR} that is rotated about a point P to produce $\overline{Q'R'}$. The three cases are shown below. The first case is proved in Example 1.

CASE 1



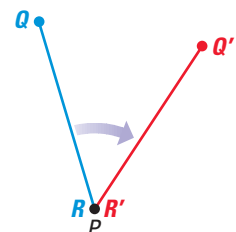
R , Q , and P are noncollinear.

CASE 2



R , Q , and P are collinear.

CASE 3



P and R are the same point.

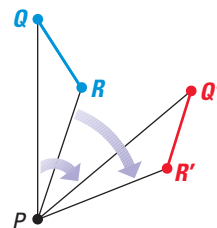


EXAMPLE 1 Proof of Theorem 7.2

Write a paragraph proof for Case 1 of the Rotation Theorem.

GIVEN ▶ A rotation about P maps Q onto Q' and R onto R' .

PROVE ▶ $\overline{QR} \cong \overline{Q'R'}$



SOLUTION

Paragraph Proof By the definition of a rotation, $PQ = PQ'$ and $PR = PR'$. Also, by the definition of a rotation, $m\angle QPQ' = m\angle RPR'$.

You can use the Angle Addition Postulate and the subtraction property of equality to conclude that $m\angle QPR = m\angle Q'PR'$. This allows you to use the SAS Congruence Postulate to conclude that $\triangle QPR \cong \triangle Q'PR'$. Because corresponding parts of congruent triangles are congruent, $\overline{QR} \cong \overline{Q'R'}$.

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STUDENT HELP

Look Back

For help with using a protractor, see p. 27.

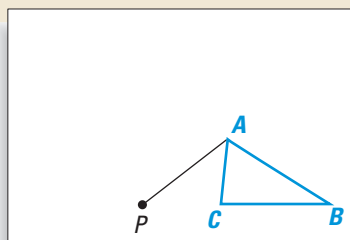
You can use a compass and a protractor to help you find the images of a polygon after a rotation. The following construction shows you how.

ACTIVITY

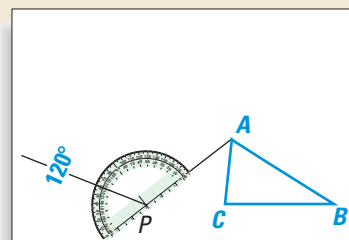
Construction

Rotating a Figure

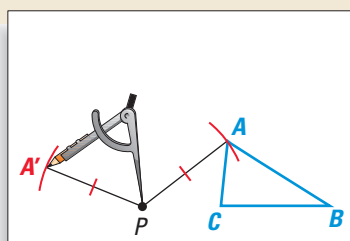
Use the following steps to draw the image of $\triangle ABC$ after a 120° counterclockwise rotation about point P .



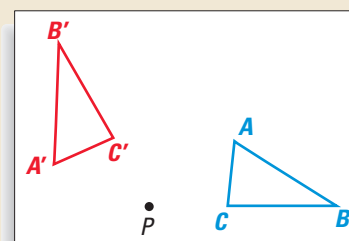
- 1 Draw a segment connecting vertex A and the center of rotation point P .



- 2 Use a protractor to measure a 120° angle counterclockwise and draw a ray.



- 3 Place the point of the compass at P and draw an arc from A to locate A' .



- 4 Repeat **Steps 1–3** for each vertex. Connect the vertices to form the image.

EXAMPLE 2 Rotations in a Coordinate Plane

In a coordinate plane, sketch the quadrilateral whose vertices are $A(2, -2)$, $B(4, 1)$, $C(5, 1)$, and $D(5, -1)$. Then, rotate $ABCD$ 90° counterclockwise about the origin and name the coordinates of the new vertices. Describe any patterns you see in the coordinates.

SOLUTION

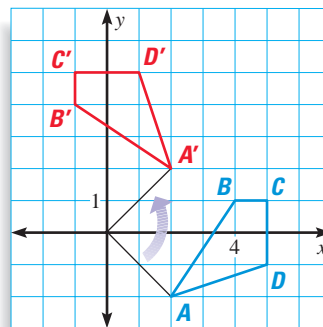
Plot the points, as shown in blue. Use a protractor, a compass, and a straightedge to find the rotated vertices. The coordinates of the preimage and image are listed below.

Figure $ABCD$

$A(2, -2)$
 $B(4, 1)$
 $C(5, 1)$
 $D(5, -1)$

Figure $A'B'C'D'$

$A'(2, 2)$
 $B'(-1, 4)$
 $C'(-1, 5)$
 $D'(1, 5)$



In the list above, the x -coordinate of the image is the opposite of the y -coordinate of the preimage. The y -coordinate of the image is the x -coordinate of the preimage.

► This transformation can be described as $(x, y) \rightarrow (-y, x)$.

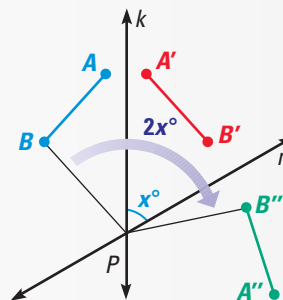
THEOREM

THEOREM 7.3

If lines k and m intersect at point P , then a reflection in k followed by a reflection in m is a rotation about point P .

The angle of rotation is $2x^\circ$, where x° is the measure of the acute or right angle formed by k and m .

$$m\angle BPB'' = 2x^\circ$$

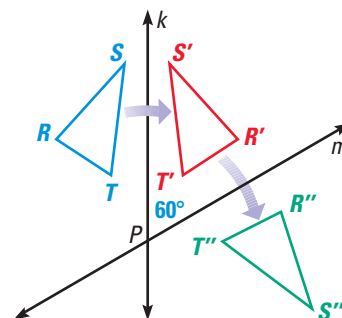


EXAMPLE 3 Using Theorem 7.3

In the diagram, $\triangle RST$ is reflected in line k to produce $\triangle R'S'T'$. This triangle is then reflected in line m to produce $\triangle R''S''T''$. Describe the transformation that maps $\triangle RST$ to $\triangle R''S''T''$.

SOLUTION

The acute angle between lines k and m has a measure of 60° . Applying Theorem 7.3 you can conclude that the transformation that maps $\triangle RST$ to $\triangle R''S''T''$ is a clockwise rotation of 120° about point P .

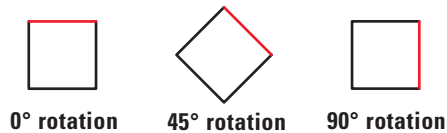


STUDENT HELP

INTERNET
HOMEWORK HELP
 Visit our Web site
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 for extra examples.

GOAL 2 ROTATIONS AND ROTATIONAL SYMMETRY

A figure in the plane has **rotational symmetry** if the figure can be mapped onto itself by a rotation of 180° or less. For instance, a square has rotational symmetry because it maps onto itself by a rotation of 90° .



EXAMPLE 4 Identifying Rotational Symmetry

Which figures have rotational symmetry? For those that do, describe the rotations that map the figure onto itself.

a. Regular octagon



b. Parallelogram



c. Trapezoid



SOLUTION

- This octagon has rotational symmetry. It can be mapped onto itself by a clockwise or counterclockwise rotation of 45° , 90° , 135° , or 180° about its center.
- This parallelogram has rotational symmetry. It can be mapped onto itself by a clockwise or counterclockwise rotation of 180° about its center.
- The trapezoid does not have rotational symmetry.

EXAMPLE 5 Using Rotational Symmetry

LOGO DESIGN A music store called Ozone is running a contest for a store logo. The winning logo will be displayed on signs throughout the store and in the store's advertisements. The only requirement is that the logo include the store's name. Two of the entries are shown below. What do you notice about them?

a.



b.



SOLUTION

- This design has rotational symmetry about its center. It can be mapped onto itself by a clockwise or counterclockwise rotation of 180° .
- This design also has rotational symmetry about its center. It can be mapped onto itself by a clockwise or counterclockwise rotation of 90° or 180° .

FOCUS ON CAREERS



LOGO DESIGNERS

create symbols that represent the name of a company or organization. The logos appear on packaging, letterheads, and Web sites.



CAREER LINK

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GUIDED PRACTICE

Vocabulary Check ✓

1. What is a *center of rotation*?

Concept Check ✓

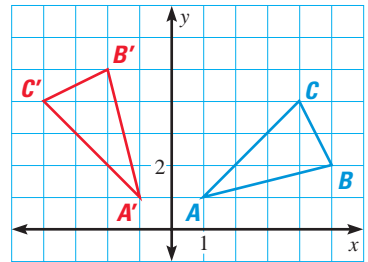
Use the diagram, in which $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by a rotation of 90° about the origin.

2. Is the rotation clockwise or counterclockwise?

3. Does $AB = A'B'$? Explain.

4. Does $AA' = BB'$? Explain.

5. If the rotation of $\triangle ABC$ onto $\triangle A'B'C'$ was obtained by a reflection of $\triangle ABC$ in some line k followed by a reflection in some line m , what would be the measure of the acute angle between lines k and m ? Explain.



Skill Check ✓

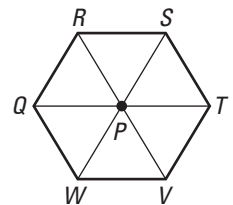
The diagonals of the regular hexagon below form six equilateral triangles. Use the diagram to complete the sentence.

6. A clockwise rotation of 60° about P maps R onto $\underline{\hspace{1cm}}?$

7. A counterclockwise rotation of 60° about $\underline{\hspace{1cm}}?$ maps R onto Q .

8. A clockwise rotation of 120° about Q maps R onto $\underline{\hspace{1cm}}?$

9. A counterclockwise rotation of 180° about P maps V onto $\underline{\hspace{1cm}}?$



Determine whether the figure has rotational symmetry. If so, describe the rotations that map the figure onto itself.



PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 816.

DESCRIBING AN IMAGE State the segment or triangle that represents the image. You can use tracing paper to help you visualize the rotation.

13. 90° clockwise rotation of \overline{AB} about P

14. 90° clockwise rotation of \overline{KF} about P

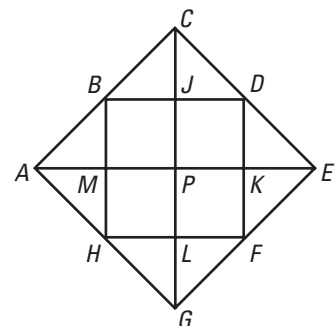
15. 90° counterclockwise rotation of \overline{CE} about E

16. 90° counterclockwise rotation of \overline{FL} about H

17. 180° rotation of $\triangle KEF$ about P

18. 180° rotation of $\triangle BCJ$ about P

19. 90° clockwise rotation of $\triangle APG$ about P



STUDENT HELP

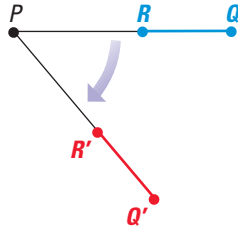
HOMEWORK HELP

- Example 1:** Exs. 13–21
- Example 2:** Exs. 22–29
- Example 3:** Exs. 30–33
- Example 4:** Exs. 36–38
- Example 5:** Exs. 39–42

PARAGRAPH PROOF Write a paragraph proof for the case of Theorem 7.2.

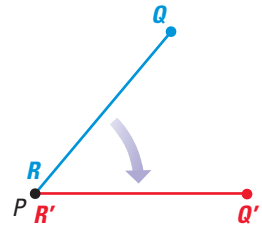
20. GIVEN ▶ A rotation about P maps Q onto Q' and R onto R' .

PROVE ▶ $\overline{QR} \cong \overline{Q'R'}$



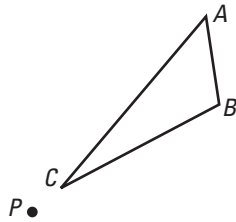
21. GIVEN ▶ A rotation about P maps Q onto Q' and R onto R' . P and R are the same point.

PROVE ▶ $\overline{QR} \cong \overline{Q'R'}$

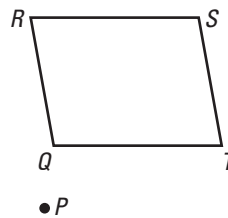


ROTATING A FIGURE Trace the polygon and point P on paper. Then, use a straightedge, compass, and protractor to rotate the polygon clockwise the given number of degrees about P .

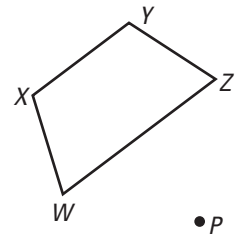
22. 60°



23. 135°

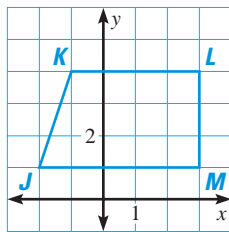


24. 150°

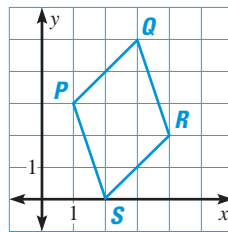


ROTATIONS IN A COORDINATE PLANE Name the coordinates of the vertices of the image after a clockwise rotation of the given number of degrees about the origin.

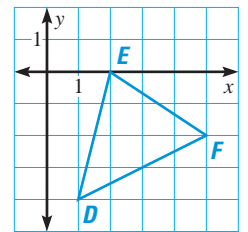
25. 90°



26. 180°

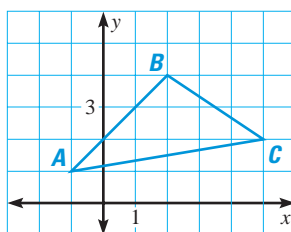


27. 270°

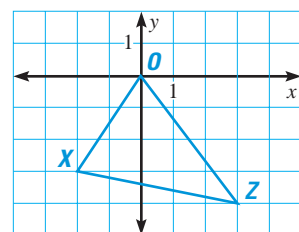


FINDING A PATTERN Use the given information to rotate the triangle. Name the vertices of the image and compare with the vertices of the preimage. Describe any patterns you see.

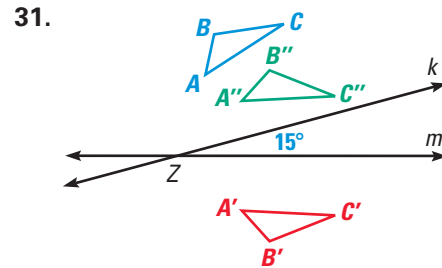
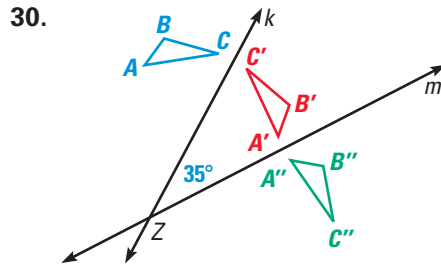
28. 90° clockwise about origin



29. 180° clockwise about origin



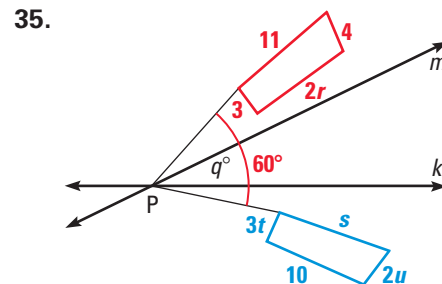
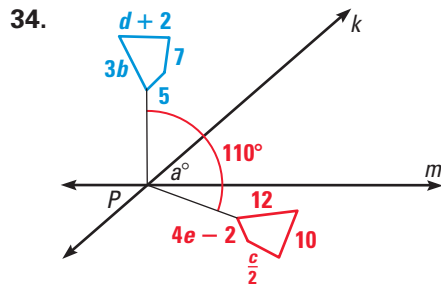
USING THEOREM 7.3 Find the angle of rotation that maps $\triangle ABC$ onto $\triangle A''B''C''$.



LOGICAL REASONING Lines m and n intersect at point D . Consider a reflection of $\triangle ABC$ in line m followed by a reflection in line n .

32. What is the angle of rotation about D , when the measure of the acute angle between lines m and n is 36° ?
33. What is the measure of the acute angle between lines m and n , when the angle of rotation about D is 162° ?

USING ALGEBRA Find the value of each variable in the rotation of the polygon about point P .

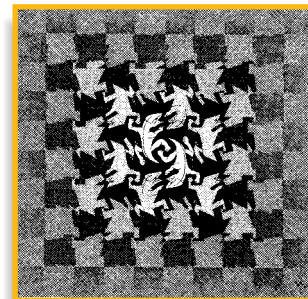


WHEEL HUBS Describe the rotational symmetry of the wheel hub.



ROTATIONS IN ART In Exercises 39–42, refer to the image below by M.C. Escher. The piece is called *Development I* and was completed in 1937.

39. Does the piece have rotational symmetry? If so, describe the rotations that map the image onto itself.
40. Would your answer to Exercise 39 change if you disregard the shading of the figures? Explain your reasoning.
41. Describe the center of rotation.
42. Is it possible that this piece could be hung upside down? Explain.



M.C. Escher's "Development I" © 1999 Cordon Art B.V. - Baarn - Holland. All rights reserved.

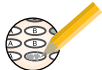
FOCUS ON PEOPLE



M.C. ESCHER is a Dutch graphic artist whose works include optical illusions and geometric patterns. (M.C. Escher's "Hand with Reflecting Sphere" © 1999 Cordon Art B.V. - Baarn - Holland. All rights reserved.)

APPLICATION LINK
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Test Preparation



43. **MULTI-STEP PROBLEM** Follow the steps below.
- Graph $\triangle RST$ whose vertices are $R(1, 1)$, $S(4, 3)$, and $T(5, 1)$.
 - Reflect $\triangle RST$ in the y -axis to obtain $\triangle R'S'T'$. Name the coordinates of the vertices of the reflection.
 - Reflect $\triangle R'S'T'$ in the line $y = -x$ to obtain $\triangle R''S''T''$. Name the coordinates of the vertices of the reflection.
 - Describe a single transformation that maps $\triangle RST$ onto $\triangle R''S''T''$.
 - Writing** Explain how to show a 90° counterclockwise rotation of any polygon about the origin using two reflections of the figure.

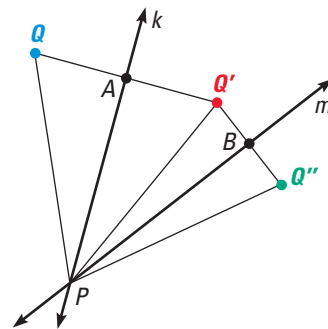
★ Challenge

44. **PROOF** Use the diagram and the given information to write a paragraph proof for Theorem 7.3.

GIVEN ▶ Lines k and m intersect at point P , Q is any point not on k or m .

PROVE ▶ a. If you reflect point Q in k , and then reflect its image Q' in m , Q'' is the image of Q after a rotation about point P .

b. $m\angle QPQ'' = 2(m\angle APB)$.



Plan for Proof First show $k \perp \overline{QQ'}$ and $\overline{QA} \cong \overline{Q'A}$. Then show $\triangle QAP \cong \triangle Q'AP$. Use a similar argument to show $\triangle Q'BP \cong \triangle Q''BP$. Use the congruent triangles and substitution to show that $\overline{QP} \cong \overline{Q''P}$. That proves part (a) by the definition of a rotation. You can use the congruent triangles to prove part (b).

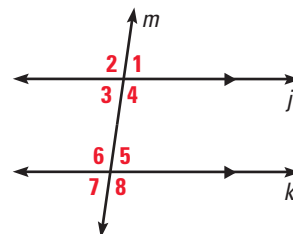
EXTRA CHALLENGE

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MIXED REVIEW

PARALLEL LINES Find the measure of the angle using the diagram, in which $j \parallel k$ and $m\angle 1 = 82^\circ$. (Review 3.3 for 7.4)

- | | |
|-----------------|-----------------|
| 45. $m\angle 5$ | 46. $m\angle 7$ |
| 47. $m\angle 3$ | 48. $m\angle 6$ |
| 49. $m\angle 4$ | 50. $m\angle 8$ |



DRAWING TRIANGLES In Exercises 51–53, draw the triangle. (Review 5.2)

- Draw a triangle whose circumcenter lies outside the triangle.
- Draw a triangle whose circumcenter lies on the triangle.
- Draw a triangle whose circumcenter lies inside the triangle.

54. **PARALLELOGRAMS** Can it be proven that the figure at the right is a parallelogram? If not, explain why not. (Review 6.2)

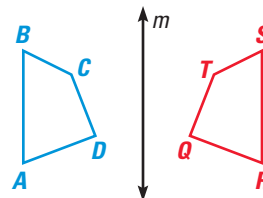


QUIZ 1

Self-Test for Lessons 7.1–7.3


Use the transformation at the right. (Lesson 7.1)

- Figure $ABCD \rightarrow$ Figure $\underline{\quad?}$
- Name and describe the transformation.
- Is the transformation an isometry? Explain.



In Exercises 4–7, find the coordinates of the reflection without using a coordinate plane. (Lesson 7.2)

- $L(2, 3)$ reflected in the x -axis
- $M(-2, -4)$ reflected in the y -axis
- $N(-4, 0)$ reflected in the x -axis
- $P(8.2, -3)$ reflected in the y -axis

-  **KNOTS** The knot at the right is a *wall knot*, which is generally used to prevent the end of a rope from running through a pulley. Describe the rotations that map the knot onto itself and describe the center of rotation. (Lesson 7.3)



MATH & History

History of Decorative Patterns



APPLICATION LINK

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THEN

FOR THOUSANDS OF YEARS, people have adorned their buildings, pottery, clothing, and jewelry with decorative patterns. Simple patterns were created by using a transformation of a shape.

NOW

TODAY, you are likely to find computer generated patterns decorating your clothes, CD covers, sports equipment, computer desktop, and even textbooks.

- The design at the right is based on a piece of pottery by Marsha Gomez. How many lines of symmetry does the design have?
- Does the design have rotational symmetry? If so, describe the rotation that maps the pattern onto itself.

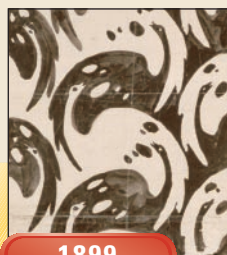
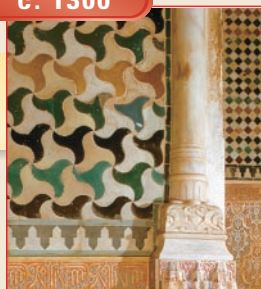


c. 1300 B.C.

Egyptian jewelry is decorated with patterns.

Tiles are arranged in symmetric patterns in the Alhambra in Spain.

c. 1300



1899

Painted textile pattern called 'Bulow Birds'

Marsha Gomez decorates pottery with symmetrical patterns.

1990s

