6.3 Proving Quadrilaterals are Parallelograms

What you should learn

GOAL 1 Prove that a quadrilateral is a parallelogram.

GOAL 2 Use coordinate geometry with parallelograms.

Why you should learn it

▼ To understand how **real-life** tools work, such as the bicycle derailleur in **Ex. 27**, which lets you change gears when you are biking uphill.

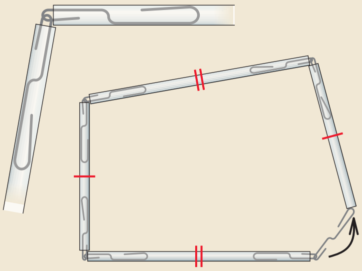
REAL LIFE

GOAL 1 PROVING QUADRILATERALS ARE PARALLELOGRAMS

The activity illustrates one way to prove that a quadrilateral is a parallelogram.

ACTIVITY Developing Concepts — Investigating Properties of Parallelograms

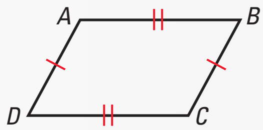
1. Cut four straws to form two congruent pairs.
2. Partly unbend two paperclips, link their smaller ends, and insert the larger ends into two cut straws, as shown. Join the rest of the straws to form a quadrilateral with opposite sides congruent, as shown.
3. Change the angles of your quadrilateral. Is your quadrilateral always a parallelogram?



THEOREMS

THEOREM 6.6

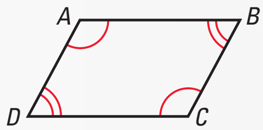
If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



***ABCD* is a parallelogram.**

THEOREM 6.7

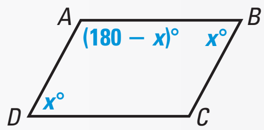
If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



***ABCD* is a parallelogram.**

THEOREM 6.8

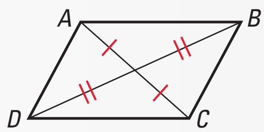
If an angle of a quadrilateral is supplementary to both of its consecutive angles, then the quadrilateral is a parallelogram.



***ABCD* is a parallelogram.**

THEOREM 6.9

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



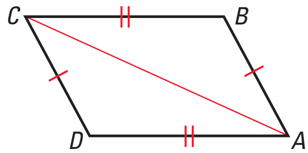
***ABCD* is a parallelogram.**

The proof of Theorem 6.6 is given in Example 1. You will be asked to prove Theorem 6.7, Theorem 6.8, and Theorem 6.9 in Exercises 32–36.

EXAMPLE 1 *Proof of Theorem 6.6*

Proof

Prove Theorem 6.6.



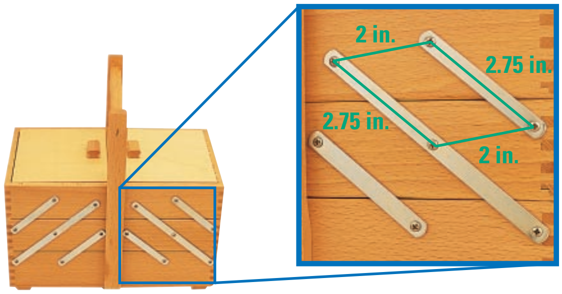
GIVEN ▶ $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$

PROVE ▶ $ABCD$ is a parallelogram.

Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$	**1.** Given
2. $\overline{AC} \cong \overline{AC}$	**2.** Reflexive Property of Congruence
3. $\triangle ABC \cong \triangle CDA$	**3.** SSS Congruence Postulate
4. $\angle BAC \cong \angle DCA$, $\angle DAC \cong \angle BCA$	**4.** Corresponding parts of $\cong$ ⚠ are $\cong$.
5. $\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{CB}$	**5.** Alternate Interior Angles Converse
6. $ABCD$ is a ▱.	**6.** Definition of parallelogram

EXAMPLE 2 *Proving Quadrilaterals are Parallelograms*

As the sewing box below is opened, the trays are always parallel to each other. Why?



FOCUS ON APPLICATIONS

REAL LIFE

CONTAINERS Many containers, such as tackle boxes, jewelry boxes, and tool boxes, use parallelograms in their design to ensure that the trays stay level.

SOLUTION

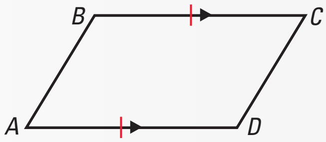
Each pair of hinges are opposite sides of a quadrilateral. The 2.75 inch sides of the quadrilateral are opposite and congruent. The 2 inch sides are also opposite and congruent. Because opposite sides of the quadrilateral are congruent, it is a parallelogram. By the definition of a parallelogram, opposite sides are parallel, so the trays of the sewing box are always parallel.

Theorem 6.10 gives another way to prove a quadrilateral is a parallelogram.

THEOREM

THEOREM 6.10

If one pair of opposite sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

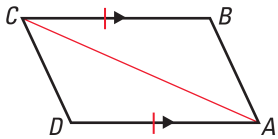


***ABCD* is a parallelogram.**

EXAMPLE 3 ***Proof of Theorem 6.10***

Proof

Prove Theorem 6.10.



GIVEN ▶ $\overline{BC} \parallel \overline{DA}$, $\overline{BC} \cong \overline{DA}$

PROVE ▶ $ABCD$ is a parallelogram.

Plan for Proof Show that $\triangle BAC \cong \triangle DCA$, so $\overline{AB} \cong \overline{CD}$. Use Theorem 6.6.

$\overline{BC} \parallel \overline{DA}$ — Given

$\angle DAC \cong \angle BCA$ — Alt. Int. ⦣ Thm.

$\overline{AC} \cong \overline{AC}$ — Refl. Prop. of Cong.

$\overline{BC} \cong \overline{DA}$ — Given

$\triangle BAC \cong \triangle DCA$ — SAS Congruence Post.

$\overline{AB} \cong \overline{CD}$ — Corresp. parts of $\cong$ ⊿ are $\cong$.

$ABCD$ is a ▱. — If opp. sides of a quad. are $\cong$, then it is a ▱.

. . . .

You have studied several ways to prove that a quadrilateral is a parallelogram. In the box below, the first way is also the definition of a parallelogram.

CONCEPT SUMMARY **PROVING QUADRILATERALS ARE PARALLELOGRAMS**

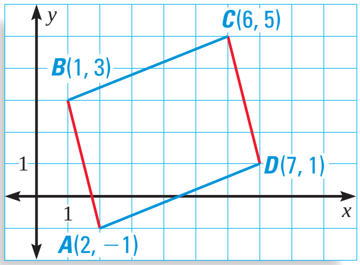
- Show that both pairs of opposite sides are parallel.
- Show that both pairs of opposite sides are congruent.
- Show that both pairs of opposite angles are congruent.
- Show that one angle is supplementary to both consecutive angles.
- Show that the diagonals bisect each other.
- Show that one pair of opposite sides are congruent and parallel.

GOAL 2 USING COORDINATE GEOMETRY

When a figure is in the coordinate plane, you can use the Distance Formula to prove that sides are congruent and you can use the slope formula to prove that sides are parallel.

EXAMPLE 4 *Using Properties of Parallelograms*

Show that $A(2, -1)$, $B(1, 3)$, $C(6, 5)$, and $D(7, 1)$ are the vertices of a parallelogram.



SOLUTION

There are many ways to solve this problem.

Method 1 Show that opposite sides have the same slope, so they are parallel.

$$\text{Slope of } \overline{AB} = \frac{3-(-1)}{1-2} = -4$$

$$\text{Slope of } \overline{CD} = \frac{1-5}{7-6} = -4$$

$$\text{Slope of } \overline{BC} = \frac{5-3}{6-1} = \frac{2}{5}$$

$$\text{Slope of } \overline{DA} = \frac{-1-1}{2-7} = \frac{2}{5}$$

$\overline{AB}$ and $\overline{CD}$ have the same slope so they are parallel. Similarly, $\overline{BC} \parallel \overline{DA}$.

▶ Because opposite sides are parallel, $ABCD$ is a parallelogram.

Method 2 Show that opposite sides have the same length.

$$AB = \sqrt{(1-2)^2 + [3-(-1)]^2} = \sqrt{17}$$

$$CD = \sqrt{(7-6)^2 + (1-5)^2} = \sqrt{17}$$

$$BC = \sqrt{(6-1)^2 + (5-3)^2} = \sqrt{29}$$

$$DA = \sqrt{(2-7)^2 + (-1-1)^2} = \sqrt{29}$$

▶ $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$. Because both pairs of opposite sides are congruent, $ABCD$ is a parallelogram.

Method 3 Show that one pair of opposite sides is congruent and parallel.

Find the slopes and lengths of $\overline{AB}$ and $\overline{CD}$ as shown in Methods 1 and 2.

$$\text{Slope of } \overline{AB} = \text{Slope of } \overline{CD} = -4$$

$$AB = CD = \sqrt{17}$$

▶ $\overline{AB}$ and $\overline{CD}$ are congruent and parallel, so $ABCD$ is a parallelogram.

STUDENT HELP

Study Tip
Because you don't know the measures of the angles of $ABCD$, you can *not* use Theorems 6.7 or 6.8 in Example 4.

STUDENT HELP

INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for extra examples.

GUIDED PRACTICE

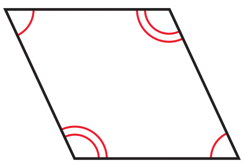
Concept Check ✓

1. Is a hexagon with opposite sides parallel called a parallelogram? Explain.

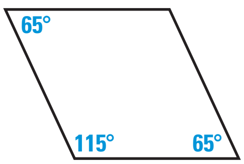
Skill Check ✓

Decide whether you are given enough information to determine that the quadrilateral is a parallelogram. Explain your reasoning.

2.



3.



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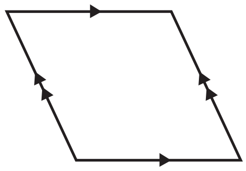
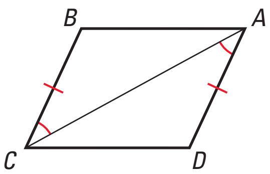


Describe how you would prove that *ABCD* is a parallelogram.

5.



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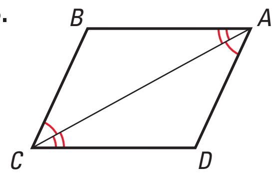

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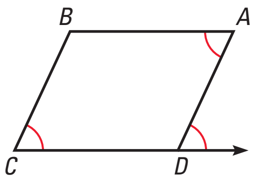


8. Describe at least three ways to show that $A(0, 0)$, $B(2, 6)$, $C(5, 7)$, and $D(3, 1)$ are the vertices of a parallelogram.

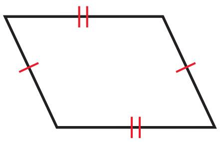
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 813.

LOGICAL REASONING **Are you given enough information to determine whether the quadrilateral is a parallelogram? Explain.**

9.



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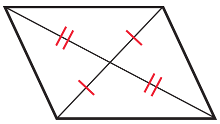

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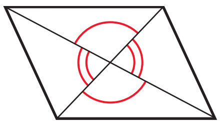

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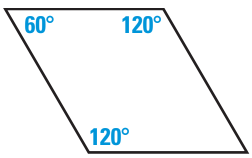

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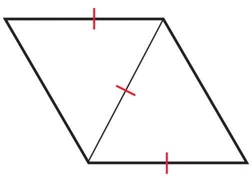

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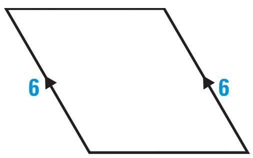


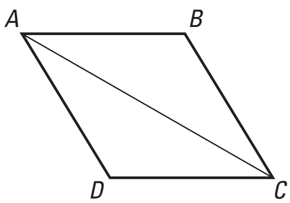
STUDENT HELP

HOMEWORK HELP

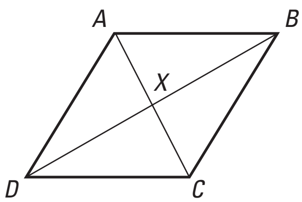
Example 1: Exs. 15, 16, 32, 33
Example 2: Exs. 21, 28, 31
Example 3: Exs. 32, 33
Example 4: Exs. 21–26, 34–36

LOGICAL REASONING **Describe how to prove that *ABCD* is a parallelogram. Use the given information.**

15. $\triangle ABC \cong \triangle CDA$

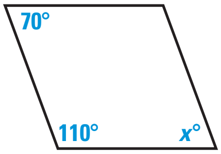


16. $\triangle AXB \cong \triangle CXD$

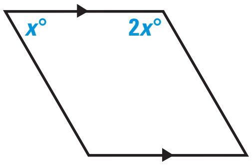


USING ALGEBRA **What value of x will make the polygon a parallelogram?**

17.



18.



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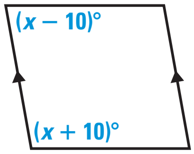


20. VISUAL THINKING Draw a quadrilateral that has one pair of congruent sides and one pair of parallel sides but that is not a parallelogram.

COORDINATE GEOMETRY **Use the given definition or theorem to prove that *ABCD* is a parallelogram. Use $A(-1, 6)$, $B(3, 5)$, $C(5, -3)$, and $D(1, -2)$.**

21. Theorem 6.6

22. Theorem 6.9

23. definition of a parallelogram

24. Theorem 6.10

USING COORDINATE GEOMETRY **Prove that the points represent the vertices of a parallelogram. Use a different method for each exercise.**

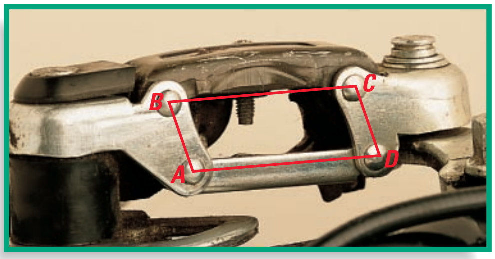
25. $J(-6, 2)$, $K(-1, 3)$, $L(2, -3)$, $M(-3, -4)$

26. $P(2, 5)$, $Q(8, 4)$, $R(9, -4)$, $S(3, -3)$

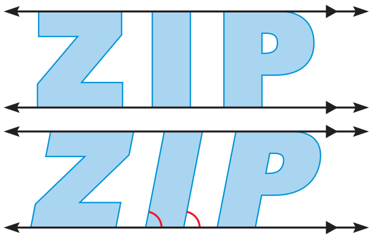
FOCUS ON APPLICATIONS

DERAILLEURS (named from the French word meaning 'derail') move the chain among two to six sprockets of different diameters to change gears.

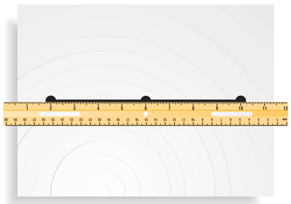
27. CHANGING GEARS When you change gears on a bicycle, the *derailleur* moves the chain to the new gear. For the derailleur at the right, $AB = 1.8$ cm, $BC = 3.6$ cm, $CD = 1.8$ cm, and $DA = 3.6$ cm. Explain why $\overline{AB}$ and $\overline{CD}$ are always parallel when the derailleur moves.



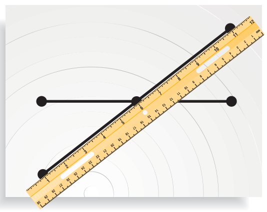
28. COMPUTERS Many word processors have a feature that allows a regular letter to be changed to an oblique (slanted) letter. The diagram at the right shows some regular letters and their oblique versions. Explain how you can prove that the oblique I is a parallelogram.



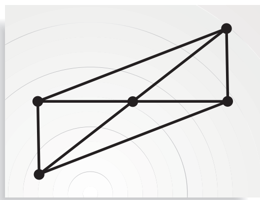
29. VISUAL REASONING Explain why the following method of drawing a parallelogram works. State a theorem to support your answer.



1. Use a ruler to draw a segment and its midpoint.



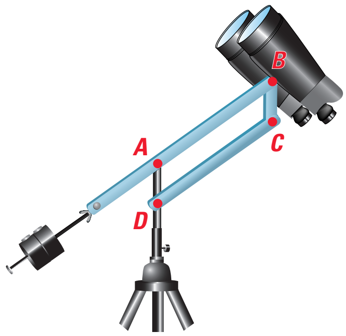
2. Draw another segment so the midpoints coincide.



3. Connect the endpoints of the segments.

30. **CONSTRUCTION** There are many ways to use a compass and straightedge to construct a parallelogram. Describe a method that uses Theorem 6.6, Theorem 6.8, or Theorem 6.10. Then use your method to construct a parallelogram.

31. **BIRD WATCHING** You are designing a binocular mount that will keep the binoculars pointed in the same direction while they are raised and lowered for different viewers. If $\overline{BC}$ is always vertical, the binoculars will always point in the same direction. How can you design the mount so $\overline{BC}$ is always vertical? Justify your answer.



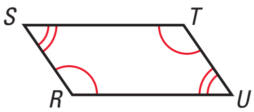
PROVING THEOREMS 6.7 AND 6.8 **Write a proof of the theorem.**

32. Prove Theorem 6.7.

GIVEN ▶ $\angle R \cong \angle T$, $\angle S \cong \angle U$

PROVE ▶ $RSTU$ is a parallelogram.

Plan for Proof Show that the sum $2(m\angle S) + 2(m\angle T) = 360°$, so $\angle S$ and $\angle T$ are supplementary and $\overline{SR} \parallel \overline{UT}$.

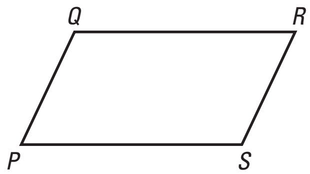


33. Prove Theorem 6.8.

GIVEN ▶ $\angle P$ is supplementary to $\angle Q$ and $\angle S$.

PROVE ▶ $PQRS$ is a parallelogram.

Plan for Proof Show that opposite sides of $PQRS$ are parallel.



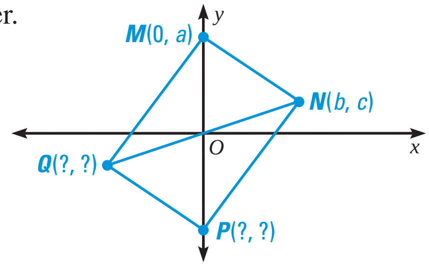
PROVING THEOREM 6.9 **In Exercises 34–36, complete the coordinate proof of Theorem 6.9.**

GIVEN ▶ Diagonals $\overline{MP}$ and $\overline{NQ}$ bisect each other.

PROVE ▶ $MNPQ$ is a parallelogram.

Plan for Proof Show that opposite sides of $MNPQ$ have the same slope.

Place $MNPQ$ in the coordinate plane so the diagonals intersect at the origin and $\overline{MP}$ lies on the y-axis. Let the coordinates of M be $(0, a)$ and the coordinates of N be (b, c). Copy the graph at the right.



34. What are the coordinates of P? Explain your reasoning and label the coordinates on your graph.

35. What are the coordinates of Q? Explain your reasoning and label the coordinates on your graph.

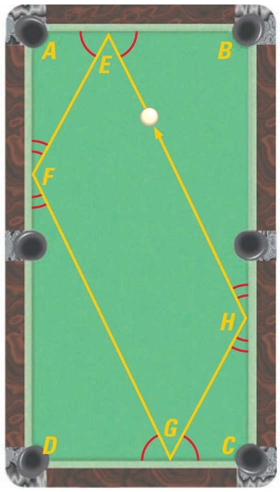
36. Find the slope of each side of $MNPQ$ and show that the slopes of opposite sides are equal.

STUDENT HELP

INTERNET
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with the coordinate proof in Exs. 34–36.

Test Preparation

37. MULTI-STEP PROBLEM You shoot a pool ball as shown at the right and it rolls back to where it started. The ball bounces off each wall at the same angle at which it hit the wall. Copy the diagram and add each angle measure as you know it.



a. The ball hits the first wall at an angle of about 63°. So $m\angle AEF = m\angle BEH \approx 63°$. Explain why $m\angle AFE \approx 27°$.

b. Explain why $m\angle FGD \approx 63°$.

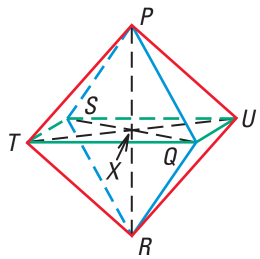
c. What is $m\angle GHC$? $m\angle EHB$?

d. Find the measure of each interior angle of $EFGH$. What kind of shape is $EFGH$? How do you know?

★ Challenge

EXTRA CHALLENGE
www.mcdougallittell.com

38. VISUAL THINKING $PQRS$ is a parallelogram and $QTSU$ is a parallelogram. Use the diagonals of the parallelograms to explain why $PTRU$ is a parallelogram.



MIXED REVIEW

USING ALGEBRA Rewrite the biconditional statement as a conditional statement and its converse. (Review 2.2 for 6.4)

39. $x^2 + 2 = 2$ if and only if $x = 0$.

40. $4x + 7 = x + 37$ if and only if $x = 10$.

41. A quadrilateral is a parallelogram if and only if each pair of opposite sides are parallel.

WRITING BICONDITIONAL STATEMENTS Write the pair of theorems from Lesson 5.1 as a single biconditional statement. (Review 2.2, 5.1 for 6.4)

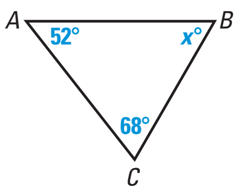
42. Theorems 5.1 and 5.2

43. Theorems 5.3 and 5.4

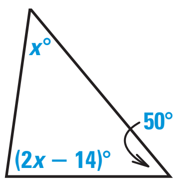
44. Write an equation of the line that is perpendicular to $y = -4x + 2$ and passes through the point $(1, -2)$. (Review 3.7)

ANGLE MEASURES Find the value of x. (Review 4.1)

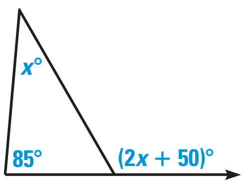
45.



46.



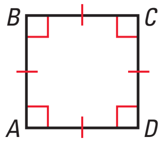
47.



QUIZ 1

Self-Test for Lessons 6.1–6.3

1. Choose the words that describe the quadrilateral at the right: *concave*, *convex*, *equilateral*, *equiangular*, and *regular*. (Lesson 6.1)

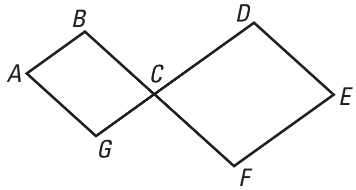


2. Find the value of x. Explain your reasoning. (Lesson 6.1)

3. Write a proof. (Lesson 6.2)

GIVEN ▶ $ABCG$ and $CDEF$ are parallelograms.

PROVE ▶ $\angle A \cong \angle E$



4. Describe two ways to show that $A(-4, 1)$, $B(3, 0)$, $C(5, -7)$, and $D(-2, -6)$ are the vertices of a parallelogram. (Lesson 6.3)

MATH & History — History of Finding Area

APPLICATION LINK www.mcdougallittell.com

THEN **THOUSANDS OF YEARS AGO,** the Egyptians needed to find the area of the land they were farming. The mathematical methods they used are described in a papyrus dating from about 1650 B.C.

NOW **TODAY,** satellites and aerial photographs can be used to measure the areas of large or inaccessible regions.

1. Find the area of the trapezoid outlined on the aerial photograph. The formula for the area of a trapezoid appears on page 374.

c.1650 B.C.

This Egyptian papyrus includes methods for finding area.

Methods for finding area are recorded in this Chinese manuscript.

c. 300 B.C.–A.D. 200

Surveyors use signals from satellites to measure large areas.

1990s