5.6 Indirect Proof and Inequalities in Two Triangles

What you should learn

GOAL 1 Read and write an indirect proof.

GOAL 2 Use the Hinge Theorem and its converse to compare side lengths and angle measures.

Why you should learn it

▼ To solve **real-life** problems, such as deciding which of two planes is farther from an airport in **Example 4** and **Exs. 28 and 29**.

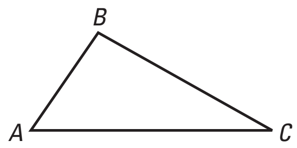
REAL LIFE

GOAL 1 USING INDIRECT PROOF

Up to now, all of the proofs in this textbook have used the Laws of Syllogism and Detachment to obtain conclusions directly. In this lesson, you will study *indirect proofs*. An **indirect proof** is a proof in which you prove that a statement is true by first assuming that its opposite is true. If this assumption leads to an impossibility, then you have proved that the original statement is true.

EXAMPLE 1 *Using Indirect Proof*

Use an indirect proof to prove that a triangle cannot have more than one obtuse angle.



SOLUTION

GIVEN ▶ $\triangle ABC$

PROVE ▶ $\triangle ABC$ does not have more than one obtuse angle.

Begin by assuming that $\triangle ABC$ *does* have more than one obtuse angle.

$m\angle A > 90°$ and $m\angle B > 90°$ — **Assume $\triangle ABC$ has two obtuse angles.**

$m\angle A + m\angle B > 180°$ — **Add the two given inequalities.**

You know, however, that the sum of the measures of all *three* angles is 180°.

$m\angle A + m\angle B + m\angle C = 180°$ — **Triangle Sum Theorem**

$m\angle A + m\angle B = 180° - m\angle C$ — **Subtraction property of equality**

So, you can substitute $180° - m\angle C$ for $m\angle A + m\angle B$ in $m\angle A + m\angle B > 180°$.

$180° - m\angle C > 180°$ — **Substitution property of equality**

$0° > m\angle C$ — **Simplify.**

The last statement is *not possible*; angle measures in triangles cannot be negative.

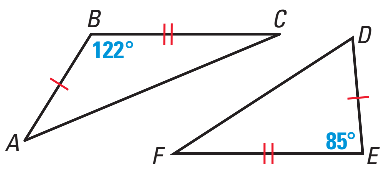
▶ So, you can conclude that the original assumption must be false. That is, $\triangle ABC$ cannot have more than one obtuse angle.

CONCEPT SUMMARY — **GUIDELINES FOR WRITING AN INDIRECT PROOF**

1. Identify the statement that you want to prove is true.
2. Begin by assuming the statement is false; assume its opposite is true.
3. Obtain statements that logically follow from your assumption.
4. If you obtain a contradiction, then the original statement must be true.

GOAL 2 USING THE HINGE THEOREM

In the two triangles shown, notice that $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, but $m\angle B$ is greater than $m\angle E$.



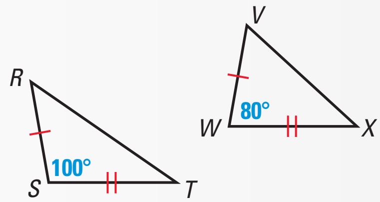
It appears that the side opposite the 122° angle is longer than the side opposite the 85° angle. This relationship is guaranteed by the Hinge Theorem below.

Exercise 31 asks you to write a proof of Theorem 5.14. Theorem 5.15 can be proved using Theorem 5.14 and indirect proof, as shown in Example 2.

THEOREMS

THEOREM 5.14 ***Hinge Theorem***

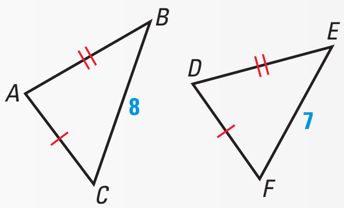
If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first is longer than the third side of the second.



$RT > VX$

THEOREM 5.15 ***Converse of the Hinge Theorem***

If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second, then the included angle of the first is larger than the included angle of the second.

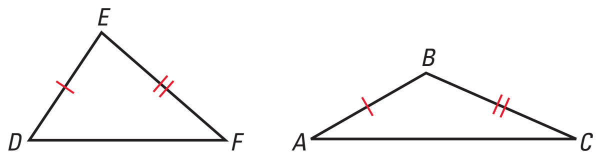


$m\angle A > m\angle D$

EXAMPLE 2 *Indirect Proof of Theorem 5.15*

GIVEN ▶ $\overline{AB} \cong \overline{DE}$
$\overline{BC} \cong \overline{EF}$
$AC > DF$

PROVE ▶ $m\angle B > m\angle E$



SOLUTION Begin by assuming that $m\angle B \ngtr m\angle E$. Then, it follows that either $m\angle B = m\angle E$ or $m\angle B < m\angle E$.

Case 1 If $m\angle B = m\angle E$, then $\angle B \cong \angle E$. So, $\triangle ABC \cong \triangle DEF$ by the SAS Congruence Postulate and $AC = DF$.

Case 2 If $m\angle B < m\angle E$, then $AC < DF$ by the Hinge Theorem.

Both conclusions contradict the given information that $AC > DF$. So the original assumption that $m\angle B \ngtr m\angle E$ cannot be correct. Therefore, $m\angle B > m\angle E$.

STUDENT HELP

Study Tip
The symbol $\ngtr$ is read as "is not greater than."

EXAMPLE 3 *Finding Possible Side Lengths and Angle Measures*

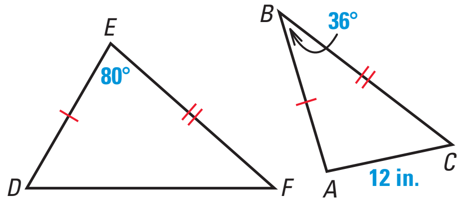
You can use the Hinge Theorem and its converse to choose possible side lengths or angle measures from a given list.

a. $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $AC = 12$ inches, $m\angle B = 36°$, and $m\angle E = 80°$. Which of the following is a possible length for $\overline{DF}$: 8 in., 10 in., 12 in., or 23 in.?

b. In a $\triangle RST$ and a $\triangle XYZ$, $\overline{RT} \cong \overline{XZ}$, $\overline{ST} \cong \overline{YZ}$, $RS = 3.7$ centimeters, $XY = 4.5$ centimeters, and $m\angle Z = 75°$. Which of the following is a possible measure for $\angle T$: 60°, 75°, 90°, or 105°?

SOLUTION

a. Because the included angle in $\triangle DEF$ is larger than the included angle in $\triangle ABC$, the third side $\overline{DF}$ must be longer than $\overline{AC}$. So, of the four choices, the only possible length for $\overline{DF}$ is 23 inches. A diagram of the triangles shows that this is plausible.



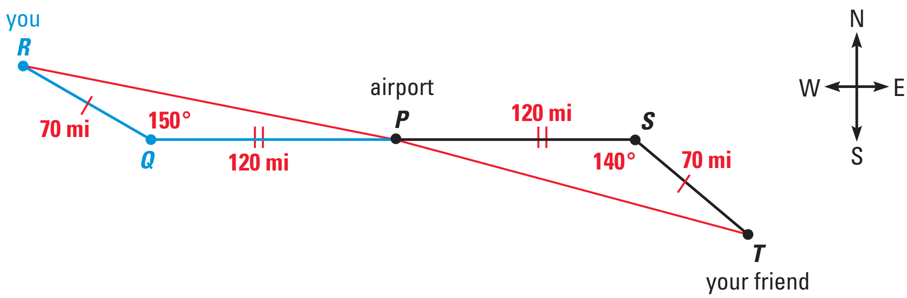
b. Because the third side in $\triangle RST$ is shorter than the third side in $\triangle XYZ$, the included angle $\angle T$ must be smaller than $\angle Z$. So, of the four choices, the only possible measure for $\angle T$ is 60°.

EXAMPLE 4 *Comparing Distances*

TRAVEL DISTANCES You and a friend are flying separate planes. You leave the airport and fly 120 miles due west. You then change direction and fly W 30° N for 70 miles. (W 30° N indicates a north-west direction that is 30° north of due west.) Your friend leaves the airport and flies 120 miles due east. She then changes direction and flies E 40° S for 70 miles. Each of you has flown 190 miles, but which plane is farther from the airport?

SOLUTION

Begin by drawing a diagram, as shown below. Your flight is represented by $\triangle PQR$ and your friend's flight is represented by $\triangle PST$.



Because these two triangles have two sides that are congruent, you can apply the Hinge Theorem to conclude that $\overline{RP}$ is longer than $\overline{TP}$.

▶ So, your plane is farther from the airport than your friend's plane.

FOCUS ON CAREERS

REAL LIFE

AIR TRAFFIC CONTROLLERS help ensure the safety of airline passengers and crews by developing air traffic flight paths that keep planes a safe distance apart.

INTERNET CAREER LINK
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GUIDED PRACTICE

Vocabulary Check ✓

1. Explain why an indirect proof might also be called a *proof by contradiction*.

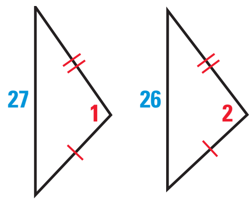
Concept Check ✓

2. To use an indirect proof to show that two lines *m* and *n* are parallel, you would first make the assumption that __?__.

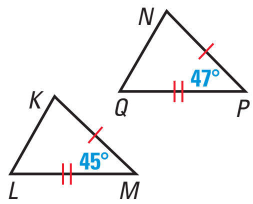
Skill Check ✓

In Exercises 3–5, complete with <, >, or =.

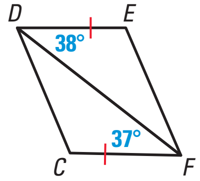
3. $m\angle 1$ __?__ $m\angle 2$



4. KL __?__ NQ



5. DC __?__ FE



6. Suppose that in a $\triangle ABC$, you want to prove that $BC > AC$. What are the two cases you would use in an indirect proof?

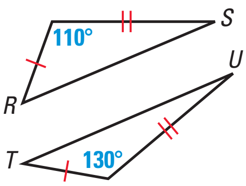
PRACTICE AND APPLICATIONS

STUDENT HELP

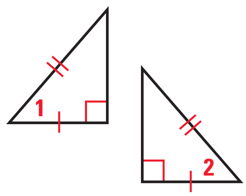
Extra Practice to help you master skills is on p. 812.

USING THE HINGE THEOREM AND ITS CONVERSE Complete with <, >, or =.

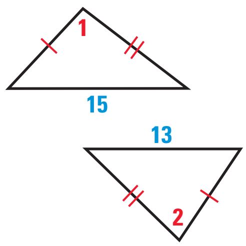
7. RS __?__ TU



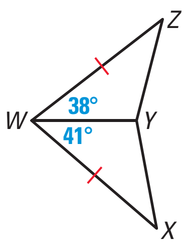
8. $m\angle 1$ __?__ $m\angle 2$



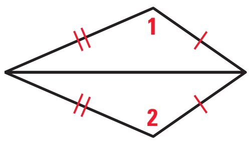
9. $m\angle 1$ __?__ $m\angle 2$



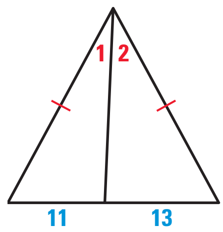
10. XY __?__ ZY



11. $m\angle 1$ __?__ $m\angle 2$



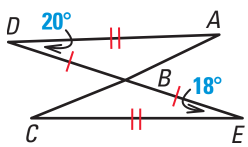
12. $m\angle 1$ __?__ $m\angle 2$



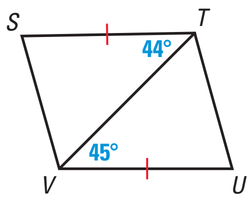
STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 21–24
Example 2: Exs. 25–27
Example 3: Exs. 7–17
Example 4: Exs. 28, 29

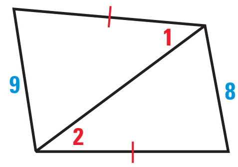
13. AB __?__ CB



14. UT __?__ SV



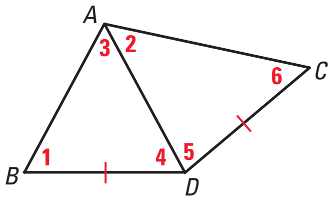
15. $m\angle 1$ __?__ $m\angle 2$



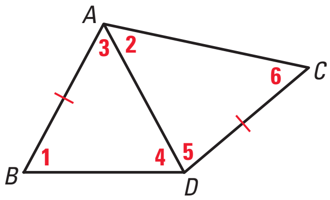
LOGICAL REASONING **In Exercises 16 and 17, match the given information with conclusion A, B, or C. Explain your reasoning.**

A. $AD > CD$ **B.** $AC > BD$ **C.** $m\angle 4 < m\angle 5$

16. $AC > AB$, $BD = CD$

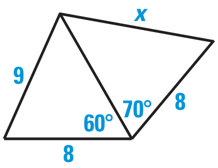


17. $AB = DC$, $m\angle 3 < m\angle 5$

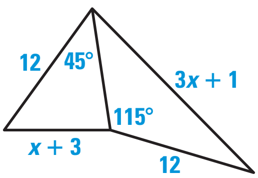


USING ALGEBRA **Use an inequality to describe a restriction on the value of *x* as determined by the Hinge Theorem or its converse.**

18.



19.



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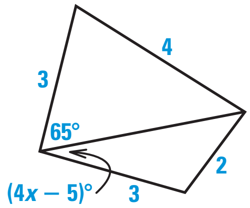


STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with negations in Exs. 21–23.

ASSUMING THE NEGATION OF THE CONCLUSION **In Exercises 21–23, write the first statement for an indirect proof of the situation.**

21. If $RS + ST \neq 12$ in. and $ST = 5$ in., then $RS \neq 7$ in.

22. In $\triangle MNP$, if Q is the midpoint of $\overline{NP}$, then $\overline{MQ}$ is a median.

23. In $\triangle ABC$, if $m\angle A + m\angle B = 90°$, then $m\angle C = 90°$.

24. **DEVELOPING PROOF** Arrange statements A–D in correct order to write an indirect proof of Postulate 7 from page 73: *If two lines intersect, then their intersection is exactly one point.*

GIVEN ▶ line *m*, line *n*

PROVE ▶ Lines *m* and *n* intersect in exactly one point.

A. But this contradicts Postulate 5, which states that there is exactly one line through any two points.

B. Then there are two lines (*m* and *n*) through points *P* and *Q*.

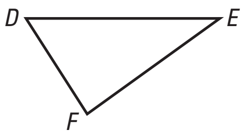
C. Assume that there are two points, *P* and *Q*, where *m* and *n* intersect.

D. It is false that *m* and *n* can intersect in two points, so they must intersect in exactly one point.

25. **PROOF** Write an indirect proof of Theorem 5.11 on page 295.

GIVEN ▶ $m\angle D > m\angle E$

PROVE ▶ $EF > DF$

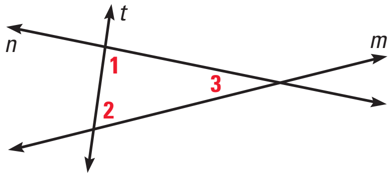


Plan for Proof In Case 1, assume that $EF < DF$. In Case 2, assume that $EF = DF$. Show that neither case can be true, so $EF > DF$.

PROOF **Write an indirect proof in paragraph form. The diagrams, which illustrate negations of the conclusions, may help you.**

26. GIVEN ▶ $\angle 1$ and $\angle 2$ are supplementary.

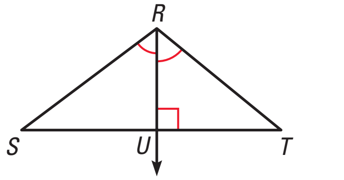
PROVE ▶ $m \parallel n$



Begin by assuming that $m \nparallel n$.

27. GIVEN ▶ $\overline{RU}$ is an altitude, $\overrightarrow{RU}$ bisects $\angle SRT$.

PROVE ▶ $\triangle RST$ is isosceles.



Begin by assuming that $RS > RT$.

COMPARING DISTANCES **In Exercises 28 and 29, consider the flight paths described. Explain how to use the Hinge Theorem to determine who is farther from the airport.**

28. *Your flight:* 100 miles due west, then 50 miles N 20° W
Friend's flight: 100 miles due north, then 50 miles N 30° E

29. *Your flight:* 210 miles due south, then 80 miles S 70° W
Friend's flight: 80 miles due north, then 210 miles N 50° E

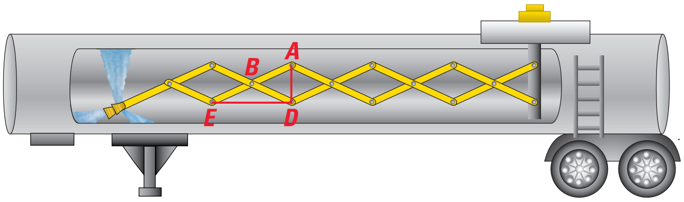
Test Preparation

30. MULTI-STEP PROBLEM Use the diagram of the tank cleaning system's expandable arm shown below.

a. As the cleaning system arm expands, $\overline{ED}$ gets longer. As ED increases, what happens to $m\angle EBD$? What happens to $m\angle DBA$?

b. Name a distance that decreases as $\overline{ED}$ gets longer.

c. *Writing* Explain how the cleaning arm illustrates the Hinge Theorem.

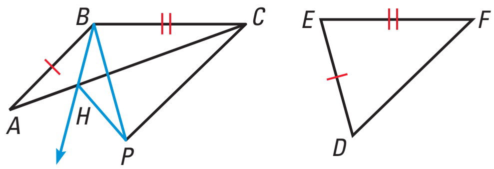


★ Challenge

31. PROOF Prove Theorem 5.14, the Hinge Theorem.

GIVEN ▶ $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $m\angle ABC > m\angle DEF$

PROVE ▶ $AC > DF$



Plan for Proof

1. Locate a point P outside $\triangle ABC$ so you can construct $\triangle PBC \cong \triangle DEF$.

2. Show that $\triangle PBC \cong \triangle DEF$ by the SAS Congruence Postulate.

3. Because $m\angle ABC > m\angle DEF$, locate a point H on $\overline{AC}$ so that $\overrightarrow{BH}$ bisects $\angle PBA$.

4. Give reasons for each equality or inequality below to show that $AC > DF$.

$$AC = AH + HC = PH + HC > PC = DF$$

EXTRA CHALLENGE
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MIXED REVIEW

CLASSIFYING TRIANGLES **State whether the triangle described is *isosceles, equiangular, equilateral,* or *scalene*.** (Review 4.1 for 6.1)

32. Side lengths: 3 cm, 5 cm, 3 cm

33. Side lengths: 5 cm, 5 cm, 5 cm

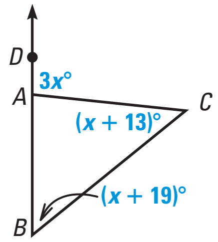
34. Side lengths: 5 cm, 6 cm, 8 cm

35. Angle measures: 30°, 30°, 120°

36. Angle measures: 60°, 60°, 60°

37. Angle measures: 65°, 50°, 65°

USING ALGEBRA **In Exercises 38–41, use the diagram shown at the right.** (Review 4.1 for 6.1)



38. Find the value of x.

39. Find $m\angle B$.

40. Find $m\angle C$.

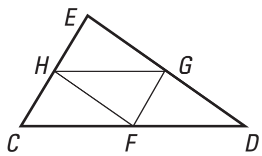
41. Find $m\angle BAC$.

42. **DESCRIBING A SEGMENT** Draw any equilateral triangle $\triangle RST$. Draw a line segment from vertex R to the midpoint of side $\overline{ST}$. State everything that you know about the line segment you have drawn. (Review 5.3)

QUIZ 2

Self-Test for Lessons 5.4–5.6

In Exercises 1–3, use the triangle shown at the right. The midpoints of the sides of $\triangle CDE$ are F, G, and H. (Lesson 5.4)



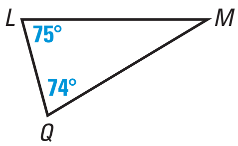
1. $\overline{FG} \parallel$ __?__

2. If $FG = 8$, then $CE =$ __?__.

3. If the perimeter of $\triangle CDE = 42$, then the perimeter of $\triangle GHF =$ __?__.

In Exercises 4–6, list the sides in order from shortest to longest. (Lesson 5.5)

4.



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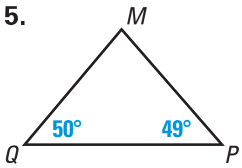

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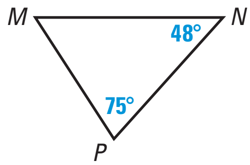
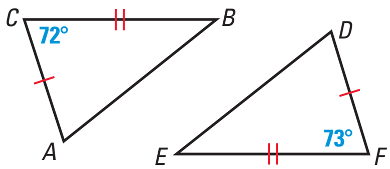
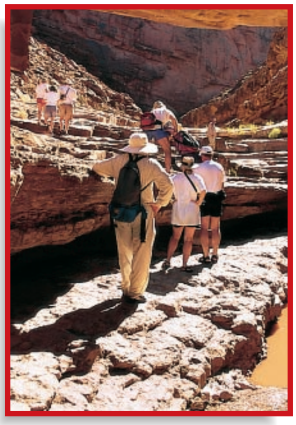


7. In $\triangle ABC$ and $\triangle DEF$ shown at the right, which is longer, $\overline{AB}$ or $\overline{DE}$? (Lesson 5.6)





Hikers in the Grand Canyon

8. **HIKING** Two groups of hikers leave from the same base camp and head in opposite directions. The first group walks 4.5 miles due east, then changes direction and walks E 45° N for 3 miles. The second group walks 4.5 miles due west, then changes direction and walks W 20° S for 3 miles. Each group has walked 7.5 miles, but which is farther from the base camp? (Lesson 5.6)