5.4 Midsegment Theorem

What you should learn

GOAL 1 Identify the midsegments of a triangle.

GOAL 2 Use properties of midsegments of a triangle.

Why you should learn it

▼ To solve **real-life** problems involving midsegments, as applied in **Exs. 32 and 35.**

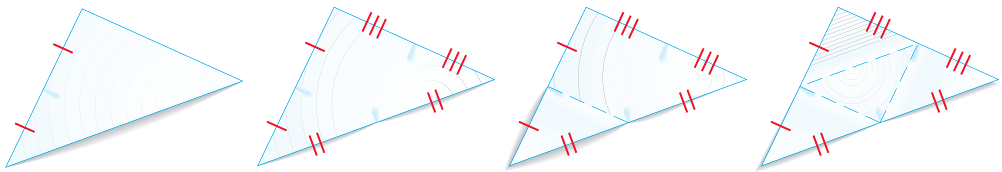
REAL LIFE

The roof of the Cowles Conservatory in Minneapolis, Minnesota, shows the midsegments of a triangle.

GOAL 1 USING MIDSEGMENTS OF A TRIANGLE

In Lessons 5.2 and 5.3, you studied four special types of segments of a triangle: perpendicular bisectors, angle bisectors, medians, and altitudes. Another special type of segment is called a *midsegment*. A **midsegment of a triangle** is a segment that connects the midpoints of two sides of a triangle.

You can form the three midsegments of a triangle by tracing the triangle on paper, cutting it out, and folding it, as shown below.

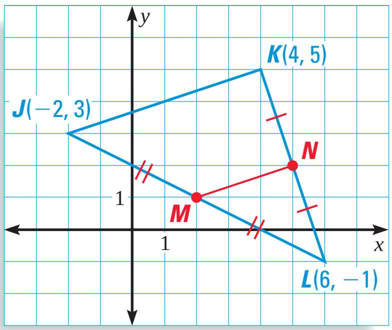


1. Fold one vertex onto another to find one midpoint.
2. Repeat the process to find the other two midpoints.
3. Fold a segment that contains two of the midpoints.
4. Fold the remaining two midsegments of the triangle.

The midsegments and sides of a triangle have a special relationship, as shown in Example 1 and Theorem 5.9 on the next page.

EXAMPLE 1 *Using Midsegments*

Show that the midsegment $\overline{MN}$ is parallel to side $\overline{JK}$ and is half as long.



SOLUTION

Use the Midpoint Formula to find the coordinates of M and N.

$$M = \left(\frac{-2+6}{2}, \frac{3+(-1)}{2}\right) = (2, 1)$$

$$N = \left(\frac{4+6}{2}, \frac{5+(-1)}{2}\right) = (5, 2)$$

Next, find the slopes of $\overline{JK}$ and $\overline{MN}$.

$$\text{Slope of } \overline{JK} = \frac{5-3}{4-(-2)} = \frac{2}{6} = \frac{1}{3} \qquad \text{Slope of } \overline{MN} = \frac{2-1}{5-2} = \frac{1}{3}$$

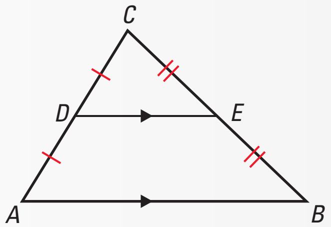
▶ Because their slopes are equal, $\overline{JK}$ and $\overline{MN}$ are parallel. You can use the Distance Formula to show that $MN = \sqrt{10}$ and $JK = \sqrt{40} = 2\sqrt{10}$. So, $\overline{MN}$ is half as long as $\overline{JK}$.

THEOREM

THEOREM 5.9 ***Midsegment Theorem***

The segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half as long.

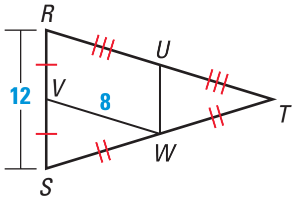
$\overline{DE} \parallel \overline{AB}$ and $DE = \frac{1}{2}AB$



EXAMPLE 2 *Using the Midsegment Theorem*

$\overline{UW}$ and $\overline{VW}$ are midsegments of $\triangle RST$. Find UW and RT.

SOLUTION



$$UW = \frac{1}{2}(RS) = \frac{1}{2}(12) = 6$$

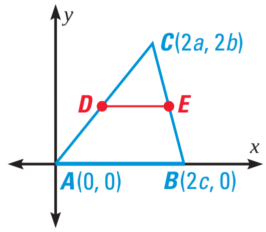
$$RT = 2(VW) = 2(8) = 16$$

A coordinate proof of Theorem 5.9 for one midsegment of a triangle is given below. Exercises 23–25 ask for proofs about the other two midsegments. To set up a coordinate proof, remember to place the figure in a convenient location.

Proof

EXAMPLE 3 *Proving Theorem 5.9*

Write a coordinate proof of the Midsegment Theorem.



SOLUTION

Place points A, B, and C in convenient locations in a coordinate plane, as shown. Use the Midpoint Formula to find the coordinates of the midpoints D and E.

$$D = \left(\frac{2a + 0}{2}, \frac{2b + 0}{2}\right) = (a, b) \qquad E = \left(\frac{2a + 2c}{2}, \frac{2b + 0}{2}\right) = (a + c, b)$$

Find the slope of midsegment $\overline{DE}$. Points D and E have the same y-coordinates, so the slope of $\overline{DE}$ is zero.

▶ $\overline{AB}$ also has a slope of zero, so the slopes are equal and $\overline{DE}$ and $\overline{AB}$ are parallel.

Calculate the lengths of $\overline{DE}$ and $\overline{AB}$. The segments are both horizontal, so their lengths are given by the absolute values of the differences of their x-coordinates.

$$AB = |2c - 0| = 2c \qquad DE = |a + c - a| = c$$

▶ The length of $\overline{DE}$ is half the length of $\overline{AB}$.

STUDENT HELP

Study Tip

In Example 3, it is convenient to locate a vertex at (0, 0) and it also helps to make one side horizontal. To use the Midpoint Formula, it is helpful for the coordinates to be multiples of 2.

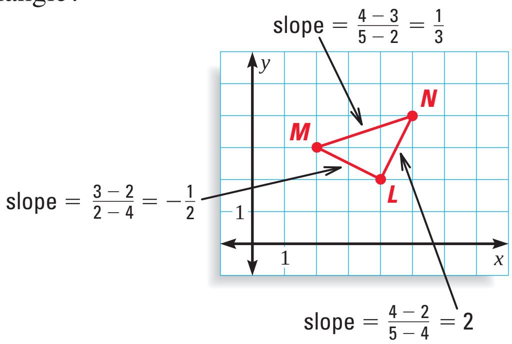
GOAL 2 USING PROPERTIES OF MIDSEGMENTS

Suppose you are given only the three midpoints of the sides of a triangle. Is it possible to draw the original triangle? Example 4 shows one method.

EXAMPLE 4 *Using Midpoints to Draw a Triangle*

Using Algebra

The midpoints of the sides of a triangle are $L(4, 2)$, $M(2, 3)$, and $N(5, 4)$. What are the coordinates of the vertices of the triangle?



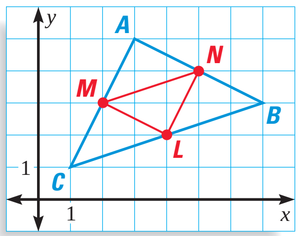
SOLUTION

Plot the midpoints in a coordinate plane.

Connect these midpoints to form the midsegments $\overline{LN}$, $\overline{MN}$, and $\overline{ML}$.

Find the slopes of the midsegments. Use the slope formula as shown.

Each midsegment contains two of the unknown triangle's midpoints and is parallel to the side that contains the third midpoint. So, you know a point on each side of the triangle and the slope of each side.



Draw the lines that contain the three sides.

▶ The lines intersect at $A(3, 5)$, $B(7, 3)$, and $C(1, 1)$, which are the vertices of the triangle.

The perimeter of the triangle formed by the three midsegments of a triangle is *half* the perimeter of the original triangle, as shown in Example 5.

EXAMPLE 5 *Perimeter of Midsegment Triangle*

FOCUS ON APPLICATIONS

REAL LIFE **ORIGAMI** is an ancient method of paper folding. The pattern of folds for a number of objects, such as the flower shown, involve midsegments.

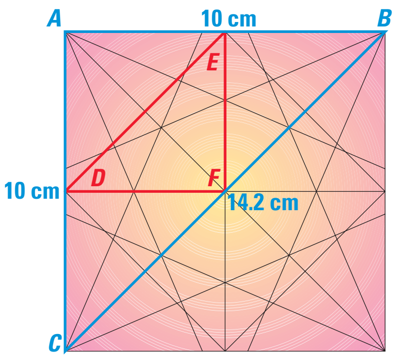
ORIGAMI $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$ are midsegments in $\triangle ABC$. Find the perimeter of $\triangle DEF$.

SOLUTION The lengths of the midsegments are half the lengths of the sides of $\triangle ABC$.

$$DF = \frac{1}{2}AB = \frac{1}{2}(10) = 5$$

$$EF = \frac{1}{2}AC = \frac{1}{2}(10) = 5$$

$$ED = \frac{1}{2}BC = \frac{1}{2}(14.2) = 7.1$$



Crease pattern of origami flower

▶ The perimeter of $\triangle DEF$ is $5 + 5 + 7.1$, or 17.1. The perimeter of $\triangle ABC$ is $10 + 10 + 14.2$, or 34.2, so the perimeter of the triangle formed by the midsegments is half the perimeter of the original triangle.

GUIDED PRACTICE

Vocabulary Check ✓

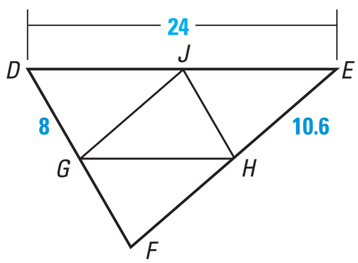
1. In $\triangle ABC$, if M is the midpoint of $\overline{AB}$, N is the midpoint of $\overline{AC}$, and P is the midpoint of $\overline{BC}$, then $\overline{MN}$, $\overline{NP}$, and $\overline{PN}$ are __?__ of $\triangle ABC$.

Concept Check ✓

2. In Example 3 on page 288, why was it convenient to position one of the sides of the triangle along the x-axis?

Skill Check ✓

In Exercises 3–9, $\overline{GH}$, $\overline{HJ}$, and $\overline{JG}$ are midsegments of $\triangle DEF$.



3. $\overline{JH} \parallel$ __?__

4. __?__ $\parallel \overline{DE}$

5. $EF =$ __?__

6. $GH =$ __?__

7. $DF =$ __?__

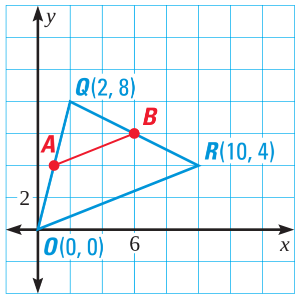
8. $JH =$ __?__

9. Find the perimeter of $\triangle GHJ$.

WALKWAYS The triangle below shows a section of walkways on a college campus.

10. The midsegment $\overline{AB}$ represents a new walkway that is to be constructed on the campus. What are the coordinates of points A and B?

11. Each unit in the coordinate plane represents 10 yards. Use the Distance Formula to find the length of the new walkway.

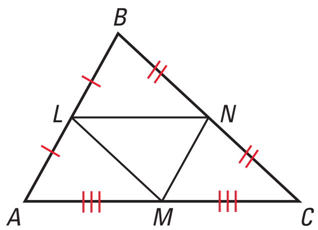


PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 812.

COMPLETE THE STATEMENT In Exercises 12–19, use $\triangle ABC$, where L, M, and N are midpoints of the sides.



12. $\overline{LM} \parallel$ __?__

13. $\overline{AB} \parallel$ __?__

14. If $AC = 20$, then $LN =$ __?__.

15. If $MN = 7$, then $AB =$ __?__.

16. If $NC = 9$, then $LM =$ __?__.

17. **USING ALGEBRA** If $LM = 3x + 7$ and $BC = 7x + 6$, then $LM =$ __?__.

18. **USING ALGEBRA** If $MN = x - 1$ and $AB = 6x - 18$, then $AB =$ __?__.

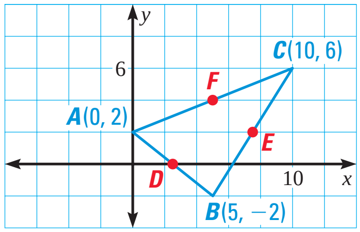
19. **LOGICAL REASONING** Which angles in the diagram are congruent? Explain your reasoning.

20. **CONSTRUCTION** Use a straightedge to draw a triangle. Then use the straightedge and a compass to construct the three midsegments of the triangle.

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 21, 22
Example 2: Exs. 12–16
Example 3: Exs. 23–25
Example 4: Exs. 26, 27
Example 5: Exs. 28, 29

USING ALGEBRA Use the diagram.



21. Find the coordinates of the endpoints of each midsegment of $\triangle ABC$.

22. Use slope and the Distance Formula to verify that the Midsegment Theorem is true for $\overline{DF}$.

USING ALGEBRA Copy the diagram in Example 3 on page 288 to complete the proof of Theorem 5.9, the Midsegment Theorem.

23. Locate the midpoint of $\overline{AB}$ and label it F. What are the coordinates of F? Draw midsegments $\overline{DF}$ and $\overline{EF}$.

24. Use slopes to show that $\overline{DF} \parallel \overline{CB}$ and $\overline{EF} \parallel \overline{CA}$.

25. Use the Distance Formula to find DF, EF, CB, and CA. Verify that $DF = \frac{1}{2}CB$ and $EF = \frac{1}{2}CA$.

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 26 and 27.

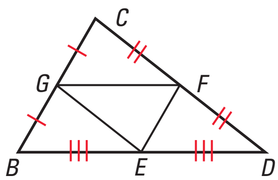
USING ALGEBRA In Exercises 26 and 27, you are given the midpoints of the sides of a triangle. Find the coordinates of the vertices of the triangle.

26. $L(1, 3)$, $M(5, 9)$, $N(4, 4)$

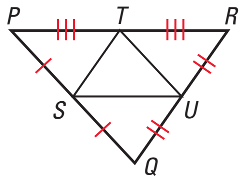
27. $L(7, 1)$, $M(9, 6)$, $N(5, 4)$

FINDING PERIMETER In Exercises 28 and 29, use the diagram shown.

28. Given $CD = 14$, $GF = 8$, and $GC = 5$, find the perimeter of $\triangle BCD$.



29. Given $PQ = 20$, $SU = 12$, and $QU = 9$, find the perimeter of $\triangle STU$.



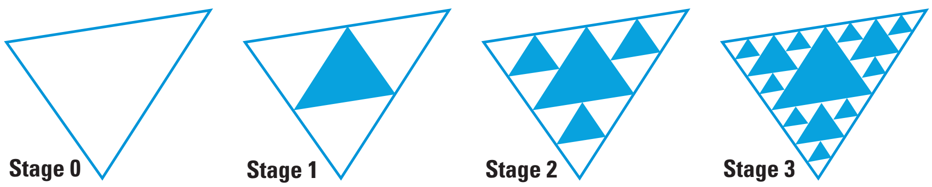
FOCUS ON APPLICATIONS

REAL LIFE FRACTALS are shapes that look the same at many levels of magnification. Take a small part of the image above and you will see that it looks about the same as the whole image.

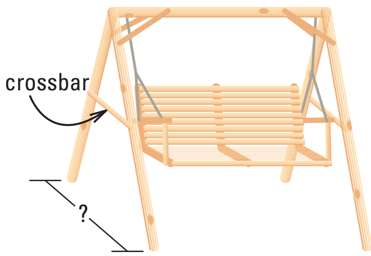
INTERNET APPLICATION LINK www.mcdougallittell.com

30. **TECHNOLOGY** Use geometry software to draw any $\triangle ABC$. Construct the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Label them as D, E, and F. Construct the midpoints of $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$. Label them as G, H, and I. What is the relationship between the perimeters of $\triangle ABC$ and $\triangle GHI$?

31. **FRACTALS** The design below, which approximates a *fractal*, is created with midsegments. Beginning with any triangle, shade the triangle formed by the three midsegments. Continue the process for each unshaded triangle. Suppose the perimeter of the original triangle is 1. What is the perimeter of the triangle that is shaded in Stage 1? What is the total perimeter of all the triangles that are shaded in Stage 2? in Stage 3?



32. **PORCH SWING** You are assembling the frame for a porch swing. The horizontal crossbars in the kit you purchased are each 30 inches long. You attach the crossbars at the midpoints of the legs. At each end of the frame, how far apart will the bottoms of the legs be when the frame is assembled? Explain.

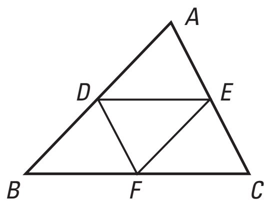


33. **WRITING A PROOF** Write a paragraph proof using the diagram shown and the given information.

GIVEN ▶ $\triangle ABC$ with midsegments $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$

PROVE ▶ $\triangle ADE \cong \triangle DBF$

Plan for Proof Use the SAS Congruence Postulate. Show that $\overline{AD} \cong \overline{DB}$. Show that because $DE = BF = \frac{1}{2}BC$, then $\overline{DE} \cong \overline{BF}$. Use parallel lines to show that $\angle ADE \cong \angle ABC$.



34. **WRITING A PLAN** Using the information from Exercise 33, write a plan for a proof showing how you could use the SSS Congruence Postulate to prove that $\triangle ADE \cong \triangle DBF$.

35. **A-FRAME HOUSE** In the A-frame house shown, the floor of the second level, labeled $\overline{PQ}$, is closer to the first floor, $\overline{RS}$, than midsegment $\overline{MN}$ is. If $\overline{RS}$ is 24 feet long, can $\overline{PQ}$ be 10 feet long? 12 feet long? 14 feet long? 24 feet long? Explain.

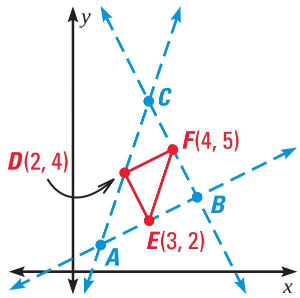
Test Preparation

36. **MULTI-STEP PROBLEM** The diagram below shows the points $D(2, 4)$, $E(3, 2)$, and $F(4, 5)$, which are midpoints of the sides of $\triangle ABC$. The directions below show how to use equations of lines to reconstruct the original $\triangle ABC$.

a. Plot D, E, and F in a coordinate plane.

b. Find the slope m_1 of one midsegment, say $\overline{DE}$.

c. The line containing side $\overline{CB}$ will have the same slope as $\overline{DE}$. Because $\overline{CB}$ contains $F(4, 5)$, an equation of $\overleftrightarrow{CB}$ in *point-slope form* is $y - 5 = m_1(x - 4)$. Write an equation of $\overleftrightarrow{CB}$.



d. Find the slopes m_2 and m_3 of the other two midsegments. Use these slopes to find equations of the lines containing the other two sides of $\triangle ABC$.

e. Rewrite your equations from parts (c) and (d) in *slope-intercept form*.

f. Use substitution to solve systems of equations to find the intersection of each pair of lines. Plot these points A, B, and C on your graph.

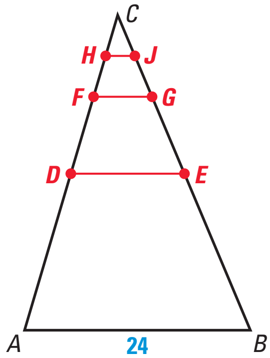
STUDENT HELP

Skills Review
For help with writing an equation of a line, see page 795.

★ **Challenge**

37. **FINDING A PATTERN** In $\triangle ABC$, the length of $\overline{AB}$ is 24. In the triangle, a succession of midsegments are formed.

- At Stage 1, draw the midsegment of $\triangle ABC$. Label it $\overline{DE}$.
- At Stage 2, draw the midsegment of $\triangle DEC$. Label it $\overline{FG}$.
- At Stage 3, draw the midsegment of $\triangle FGC$. Label it $\overline{HJ}$.



Copy and complete the table showing the length of the midsegment at each stage.

Stage n	0	1	2	3	4	5
Midsegment length	24	?	?	?	?	?

38. **USING ALGEBRA** In Exercise 37, let y represent the length of the midsegment at Stage n. Construct a scatter plot for the data given in the table. Then find a function that gives the length of the midsegment at Stage n.

EXTRA CHALLENGE
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MIXED REVIEW

SOLVING EQUATIONS **Solve the equation and state a reason for each step.** (Review 2.4)

39. $x - 3 = 11$

40. $3x + 13 = 46$

41. $8x - 1 = 2x + 17$

42. $5x + 12 = 9x - 4$

43. $2(4x - 1) = 14$

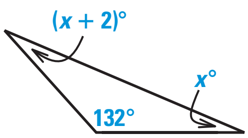
44. $9(3x + 10) = 27$

45. $-2(x + 1) + 3 = 23$

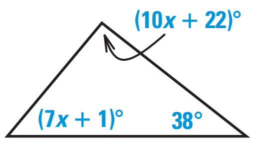
46. $3x + 2(x + 5) = 40$

USING ALGEBRA **Find the value of x.** (Review 4.1 for 5.5)

47.



48.



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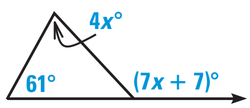

ANGLE BISECTORS $\overrightarrow{AD}$, $\overrightarrow{BD}$, **and** $\overrightarrow{CD}$ **are angle bisectors of** $\triangle ABC$. (Review 5.2)

50. Explain why $\angle CAD \cong \angle BAD$ and $\angle BCD \cong \angle ACD$.

51. Is point D the *circumcenter* or *incenter* of $\triangle ABC$?

52. Explain why $\overline{DE} \cong \overline{DG} \cong \overline{DF}$.

53. Suppose $CD = 10$ and $EC = 8$. Find DF.