Additional Proofs

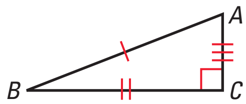
PROOF OF THEOREM 4.8
HYPOTENUSE-LEG (HL) CONGRUENCE THEOREM

THEOREM 4.8
page 238

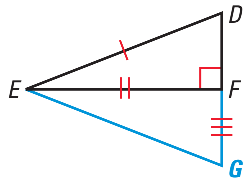
If the hypotenuse and a leg of a right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the two triangles are congruent.

GIVEN ▶ In $\triangle ABC$, $\angle C$ is a right angle.
In $\triangle DEF$, $\angle F$ is a right angle.
$\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$

PROVE ▶ $\triangle ABC \cong \triangle DEF$



Plan for Proof Construct $\triangle GEF$ with $\overline{GF} \cong \overline{AC}$, as shown. Prove that $\triangle ABC \cong \triangle GEF$. Then use the fact that corresponding parts of congruent triangles are congruent to show that $\triangle GEF \cong \triangle DEF$. By the transitive property of congruence, you can show that $\triangle ABC \cong \triangle DEF$.



Statements	Reasons
1. $\angle C$ is a right angle. $\angle DFE$ is a right angle.	**1.** Given
2. $\overline{EF} \perp \overline{DG}$	**2.** Definition of perpendicular lines
3. $\angle EFG$ is a right angle.	**3.** If 2 lines are ⊥, then they form 4 rt. ∠s.
4. $\angle C \cong \angle EFG$	**4.** Right Angles Congruence Theorem
5. $\overline{BC} \cong \overline{EF}$	**5.** Given
6. $\overline{AC} \cong \overline{GF}$	**6.** Given by construction
7. $\triangle ABC \cong \triangle GEF$	**7.** SAS Congruence Postulate
8. $\overline{GE} \cong \overline{AB}$	**8.** Corresp. parts of ≅ △s are ≅.
9. $\overline{AB} \cong \overline{DE}$	**9.** Given
10. $\overline{GE} \cong \overline{DE}$	**10.** Transitive Property of Congruence
11. $\angle D \cong \angle G$	**11.** If 2 sides of a △ are ≅, then the ∠s opposite them are ≅.
12. $\angle GFE \cong \angle DFE$	**12.** Right Angles Congruence Theorem
13. $\triangle GEF \cong \triangle DEF$	**13.** AAS Congruence Theorem
14. $\triangle ABC \cong \triangle DEF$	**14.** Transitive Prop. of ≅ △s

ANOTHER PROOF OF THEOREM 4.8
HYPOTENUSE-LEG (HL) CONGRUENCE THEOREM

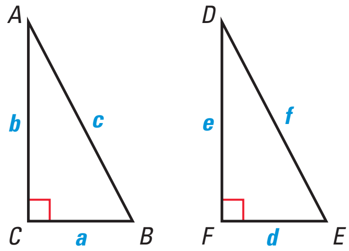
STUDENT HELP

Study Tip
This second proof of the HL Theorem uses the Pythagorean Theorem, which is introduced in Chapter 1 and further developed in Chapter 9.

GIVEN ▶ $\triangle ABC$ and $\triangle DEF$ are right triangles; $c = f$, $b = e$

PROVE ▶ $\triangle ABC \cong \triangle DEF$

Plan for Proof Use the Pythagorean Theorem to show that $a = d$. Then use the SSS Congruence Postulate.



Statements	Reasons
1. $\triangle ABC$ and $\triangle DEF$ are right triangles.	**1.** Given
2. $c = f$, $b = e$	**2.** Given
3. $c^2 = f^2$, $b^2 = e^2$	**3.** A property of squares
4. $a^2 + b^2 = c^2$; $d^2 + e^2 = f^2$	**4.** Pythagorean Theorem
5. $a^2 + b^2 = d^2 + e^2$	**5.** Substitution property of equality
6. $a^2 + e^2 = d^2 + e^2$	**6.** Substitution property of equality
7. $a^2 = d^2$	**7.** Subtraction property of equality
8. $a = d$	**8.** A property of square roots
9. $\triangle ABC \cong \triangle DEF$	**9.** SSS Congruence Postulate

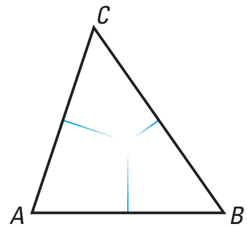
PROOF OF THEOREM 5.5
CONCURRENCY OF PERPENDICULAR BISECTORS OF A TRIANGLE

THEOREM 5.5
page 273

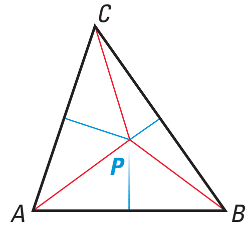
The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangle.

GIVEN ▶ $\triangle ABC$; the $\perp$ bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$

PROVE ▶ The $\perp$ bisectors intersect in a point; that point is equidistant from A, B, and C.



Plan for Proof Show that P, the point of intersection of the perpendicular bisectors of $\overline{BC}$ and $\overline{AC}$, also lies on the perpendicular bisector of $\overline{AB}$. Then show that P is equidistant from the vertices of the triangle, A, B, and C.



Statements	Reasons
1. The perpendicular bisectors of $\overline{BC}$ and $\overline{AC}$ intersect at some point P.	**1.** ABC is a triangle, so its sides $\overline{BC}$ and $\overline{AC}$ cannot be parallel; therefore, segments perpendicular to those sides cannot be parallel. So, the perpendicular bisectors must intersect in some point. Call it P.
2. Draw $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$.	**2.** Through any two points there is exactly one line.
3. $PA = PC$, $PC = PB$	**3.** If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment. (Theorem 5.1)
4. $PA = PB$	**4.** Substitution property of equality
5. P is on the perpendicular bisector of $\overline{AB}$.	**5.** If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment. (Theorem 5.2)
6. $PA = PB = PC$, so P is equidistant from the vertices of the triangle.	**6.** Steps 3 and 4 and definition of equidistant

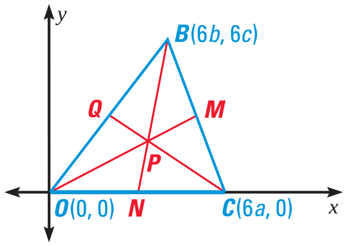
PROOF OF THEOREM 5.7
CONCURRENCY OF MEDIANS OF A TRIANGLE

THEOREM 5.7
page 279

The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.

GIVEN ▶ $\triangle OBC$; medians $\overline{OM}$, $\overline{BN}$, and $\overline{CQ}$

PROVE ▶ The medians intersect in a point P; that point is two thirds of the distance from vertices O, B, and C to midpoints M, N, and Q.



Plan for Proof The medians $\overline{OM}$ and $\overline{BN}$ intersect at some point P. Show that point P lies on $\overleftrightarrow{CQ}$. Then show that $OP = \frac{2}{3}OM$, $BP = \frac{2}{3}BN$, and $CP = \frac{2}{3}CQ$.

1 ***Find*** the equations of the medians $\overline{OM}$, $\overline{BN}$, and $\overline{CQ}$.

By the *Midpoint Formula*,

the coordinates of M are $\left(\frac{6b + 6a}{2}, \frac{6c + 0}{2}\right) = (3b + 3a, 3c)$;

the coordinates of N are $\left(\frac{0 + 6a}{2}, \frac{0 + 0}{2}\right) = (3a, 0)$;

the coordinates of Q are $\left(\frac{6b + 0}{2}, \frac{6c + 0}{2}\right) = (3b, 3c)$.

By the *slope formula*,

slope of $\overline{OM} = \frac{3c - 0}{(3b + 3a) - 0} = \frac{3c}{3b + 3a} = \frac{c}{b + a}$;

slope of $\overline{BN} = \frac{6c - 0}{6b - 3a} = \frac{6c}{6b - 3a} = \frac{2c}{2b - a}$;

slope of $\overline{CQ} = \frac{0 - 3c}{6a - 3b} = \frac{-3c}{6a - 3b} = \frac{-c}{2a - b} = \frac{c}{b - 2a}$.

Using the *point-slope form of an equation of a line*,

the equation of $\overleftrightarrow{OM}$ is $y - 0 = \frac{c}{b + a}(x - 0)$, or $y = \frac{c}{b + a}x$;

the equation of $\overleftrightarrow{BN}$ is $y - 0 = \frac{2c}{2b - a}(x - 3a)$, or $y = \frac{2c}{2b - a}(x - 3a)$;

the equation of $\overleftrightarrow{CQ}$ is $y - 0 = \frac{c}{b - 2a}(x - 6a)$, or $y = \frac{c}{b - 2a}(x - 6a)$.

2 ***Find*** the coordinates of the point P where two medians (say, $\overline{OM}$ and $\overline{BN}$) intersect. Using the substitution method, set the values of y in the equations of $\overleftrightarrow{OM}$ and $\overleftrightarrow{BN}$ equal to each other:

$$\frac{c}{b + a}x = \frac{2c}{2b - a}(x - 3a)$$
$$cx(2b - a) = 2c(x - 3a)(b + a)$$
$$2cxb - cxa = 2cxb + 2cxa - 6cab - 6ca^2$$
$$-3cxa = -6cab - 6ca^2$$
$$x = 2b + 2a$$

Substituting to find y, $y = \frac{c}{b + a}x = \frac{c}{b + a}(2b + 2a) = 2c$.

So, the coordinates of P are $(2b + 2a, 2c)$.

STUDENT HELP

Study Tip

Because you want to prove something involving the fraction $\frac{2}{3}$, it is helpful to position the vertices at points whose coordinates are multiples of both 2 and 3.

3 ***Show*** that P is on $\overleftrightarrow{CQ}$.

Substituting the x coordinate for P into the equation of $\overleftrightarrow{CQ}$,
$y = \frac{c}{b - 2a}([2b + 2a] - 6a) = \frac{c}{b - 2a}(2b - 4a) = 2c$. So, $P(2b + 2a, 2c)$ is on $\overleftrightarrow{CQ}$ and the three medians intersect at the same point.

4 ***Find*** the distances OM, OP, BN, BP, CQ, and CP.
Use the *Distance Formula*.

$$OM = \sqrt{((3b + 3a) - 0)^2 + (3c - 0)^2} = \sqrt{(3(b + a))^2 + (3c)^2} = \sqrt{9((b + a)^2 + c^2)} = 3\sqrt{(b + a)^2 + c^2}$$

$$OP = \sqrt{((2b + 2a) - 0)^2 + (2c - 0)^2} = \sqrt{(2(b + a))^2 + (2c)^2} = \sqrt{4((b + a)^2 + c^2)} = 2\sqrt{(b + a)^2 + c^2}$$

$$BN = \sqrt{(3a - 6b)^2 + (0 - 6c)^2} = \sqrt{(3a - 6b)^2 + (-6c)^2} = \sqrt{(3(a - 2b))^2 + (3(-2c))^2} = \sqrt{9(a - 2b)^2 + 9(4c^2)} = \sqrt{9((a - 2b)^2 + 4c^2)} = 3\sqrt{(a - 2b)^2 + 4c^2}$$

$$BP = \sqrt{((2b + 2a) - 6b)^2 + (2c - 6c)^2} = \sqrt{(2a - 4b)^2 + (-4c)^2} = \sqrt{(2(a - 2b))^2 + (2(-2c))^2} = \sqrt{4(a - 2b)^2 + 4(4c^2)} = \sqrt{4((a - 2b)^2 + 4c^2)} = 2\sqrt{(a - 2b)^2 + 4c^2}$$

$$CQ = \sqrt{(6a - 3b)^2 + (0 - 3c)^2} = \sqrt{(3(2a - b))^2 + (-3c)^2} = \sqrt{9((2a - b)^2 + c^2)} = 3\sqrt{(2a - b)^2 + c^2}$$

$$CP = \sqrt{(6a - (2b + 2a))^2 + (0 - 2c)^2} = \sqrt{(4a - 2b)^2 + (-2c)^2} = \sqrt{(2(2a - b))^2 + 4c^2} = \sqrt{4((2a - b)^2 + c^2)} = 2\sqrt{(2a - b)^2 + c^2}$$

5 ***Multiply*** OM, BN, and CQ by $\frac{2}{3}$.

$$\frac{2}{3}OM = \frac{2}{3}(3\sqrt{(b + a)^2 + c^2})$$
$$= 2\sqrt{(b + a)^2 + c^2}$$

$$\frac{2}{3}BN = \frac{2}{3}(3\sqrt{(a - 2b)^2 + 4c^2})$$
$$= 2\sqrt{(a - 2b)^2 + 4c^2}$$

$$\frac{2}{3}CQ = \frac{2}{3}(3\sqrt{(2a - b)^2 + c^2})$$
$$= 2\sqrt{(2a - b)^2 + c^2}$$

Thus, $OP = \frac{2}{3}OM$, $BP = \frac{2}{3}BN$, and $CP = \frac{2}{3}CQ$.

PROOF OF THEOREM 5.8
CONCURRENCY OF ALTITUDES OF A TRIANGLE

THEOREM 5.8
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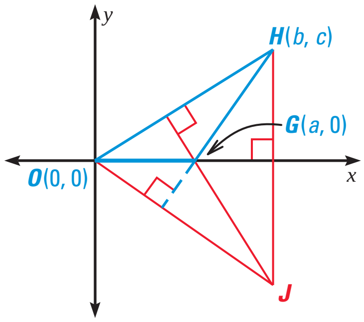
The lines containing the altitudes of a triangle are concurrent.

STUDENT HELP

Study Tip
Choose a general triangle, with one vertex at the origin and one side along an axis. In the proof shown, the triangle is obtuse.

GIVEN ▶ $\triangle OGH$

PROVE ▶ The altitudes to the sides of $\triangle OGH$ all intersect at J.



Plan for Proof Find the equations of the lines containing the altitudes of $\triangle OGH$. Find the intersection point of two of these lines. Show that the intersection point is also on the line containing the third altitude.

1 ***Find*** the slopes of the lines containing the sides $\overline{OH}$, $\overline{GH}$, and $\overline{OG}$.

slope of $\overleftrightarrow{OH} = \frac{c}{b}$ slope of $\overleftrightarrow{GH} = \frac{c}{b-a}$ slope of $\overleftrightarrow{OG} = 0$

2 ***Use*** the *Slopes of Perpendicular Lines Postulate* to find the slopes of the lines containing the altitudes.

slope of line containing altitude to $\overline{OH} = \frac{-b}{c}$

slope of line containing altitude to $\overline{GH} = \frac{-(b-a)}{c} = \frac{a-b}{c}$

The line containing the altitude to $\overline{OG}$ has an undefined slope.

3 ***Use*** the *point-slope form of an equation of a line* to write equations for the lines containing the altitudes.

An equation of the line containing the altitude to $\overline{OH}$ is

$$y - 0 = \frac{-b}{c}(x - a), \text{ or } y = \frac{-b}{c}x + \frac{ab}{c}.$$

An equation of the line containing the altitude to $\overline{GH}$ is

$$y - 0 = \frac{a-b}{c}(x - 0), \text{ or } y = \frac{a-b}{c}x.$$

An equation of the vertical line containing the altitude to $\overline{OG}$ is $x = b$.

4 ***Find*** the coordinates of the point J where the lines containing two of the altitudes intersect. Using substitution, set the values of y in two of the above equations equal to each other, then solve for x:

$$\frac{-b}{c}x + \frac{ab}{c} = \frac{a-b}{c}x$$
$$\frac{ab}{c} = \frac{a-b}{c}x + \frac{b}{c}x$$
$$\frac{ab}{c} = \frac{a}{c}x$$
$$x = b$$

Next, substitute to find y: $y = \frac{-b}{c}x + \frac{ab}{c} = \frac{-b}{c}(b) + \frac{ab}{c} = \frac{ab - b^2}{c}$.

So, the coordinates of J are $\left(b, \frac{ab - b^2}{c}\right)$.

5 ***Show*** that J is on the line that contains the altitude to side $\overline{OG}$. J is on the vertical line with equation $x = b$ because its x-coordinate is b. Thus, the lines containing the altitudes of $\triangle OGH$ are concurrent.

PROOF OF THEOREM 6.17
MIDSEGMENT THEOREM FOR TRAPEZOIDS

THEOREM 6.17
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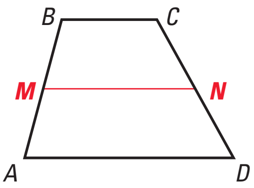
The midsegment of a trapezoid is parallel to each base and its length is one half the sum of the lengths of the bases.

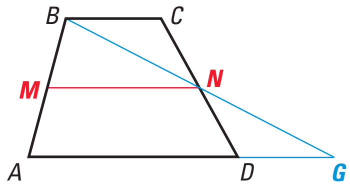
GIVEN ▶ Trapezoid $ABCD$ with midsegment $\overline{MN}$

PROVE ▶ $\overline{MN} \parallel \overline{AD}$, $\overline{MN} \parallel \overline{BC}$,

$MN = \frac{1}{2}(AD + BC)$



Plan for Proof Draw $\overline{BN}$, then extend $\overline{BN}$ and $\overline{AD}$ so that they intersect at point G. Then prove that $\triangle BNC \cong \triangle GND$, and use the fact that $\overline{MN}$ is a midsegment of $\triangle BAG$ to prove $MN = \frac{1}{2}(AD + BC)$.



Statements	Reasons
1. $ABCD$ is a trapezoid with midsegment $\overline{MN}$.	**1.** Given
2. Draw $\overline{BN}$, then extend $\overline{BN}$ and $\overline{AD}$ so that they intersect at point G.	**2.** Through any two points there is exactly one line.
3. N is the midpoint of $\overline{CD}$.	**3.** Definition of midsegment of a trapezoid
4. $\overline{CN} \cong \overline{ND}$	**4.** Definition of midpoint
5. $\overline{AD} \parallel \overline{BC}$	**5.** Definition of trapezoid
6. $\angle BCN \cong \angle GDN$	**6.** Alternate Interior ∠s Theorem
7. $\angle BNC \cong \angle GND$	**7.** Vertical angles are congruent.
8. $\triangle BNC \cong \triangle GND$	**8.** ASA Congruence Postulate
9. $\overline{BN} \cong \overline{GN}$	**9.** Corresp. parts of $\cong$ △s are $\cong$.
10. N is the midpoint of $\overline{BG}$.	**10.** Definition of midpoint
11. $\overline{MN}$ is the midsegment of $\triangle BGA$.	**11.** Definition of midsegment of a $\triangle$
12. $\overline{MN} \parallel \overline{AG}$ (so $\overline{MN} \parallel \overline{AD}$)	**12.** Midsegment of a $\triangle$ Theorem
13. $\overline{MN} \parallel \overline{BC}$	**13.** Two lines $\parallel$ to the same line are $\parallel$.
14. $MN = \frac{1}{2}AG$	**14.** Midsegment of a $\triangle$ Theorem
15. $AG = AD + DG$	**15.** Segment Addition Postulate
16. $\overline{DG} \cong \overline{BC}$	**16.** Corresp. parts of $\cong$ △s are $\cong$.
17. $DG = BC$	**17.** Definition of congruent segments
18. $AG = AD + BC$	**18.** Substitution property of equality
19. $MN = \frac{1}{2}(AD + BC)$	**19.** Substitution property of equality

PROOF OF THEOREM 10.11
A THEOREM ABOUT INSCRIBED QUADRILATERALS

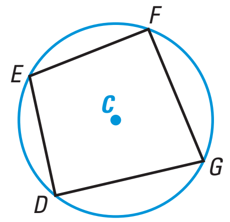
THEOREM 10.11
page 615

A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.

1 Prove that if a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

GIVEN ▶ $DEFG$ is inscribed in $\odot C$.

PROVE ▶ $\angle D$ and $\angle F$ are supplementary, $\angle E$ and $\angle G$ are supplementary.



Paragraph Proof Arcs $\overset{\frown}{EFG}$ and $\overset{\frown}{GDE}$ together make a circle, so $m\overset{\frown}{EFG} + m\overset{\frown}{GDE} = 360°$ by the Arc Addition Postulate. $\angle D$ is inscribed in $\overset{\frown}{EFG}$ and $\angle F$ is inscribed in $\overset{\frown}{GDE}$, so the angle measures are half the arc measures. Using the Substitution and Distributive properties, the sum of the measures of the opposite angles is

$$m\angle D + m\angle F = \frac{1}{2}m\overset{\frown}{EFG} + \frac{1}{2}m\overset{\frown}{GDE} = \frac{1}{2}(m\overset{\frown}{EFG} + m\overset{\frown}{GDE}) = \frac{1}{2}(360°) = 180°.$$

So, $\angle D$ and $\angle F$ are supplementary by definition. Similarly, $\angle E$ and $\angle G$ are inscribed in $\overset{\frown}{FGD}$ and $\overset{\frown}{DEF}$ and $m\angle E + m\angle G = 180°$. Then $\angle E$ and $\angle G$ are supplementary by definition.

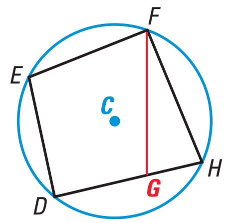
2 Prove that if the opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle.

GIVEN ▶ $\angle E$ and $\angle G$ are supplementary (or $\angle D$ and $\angle F$ are supplementary).

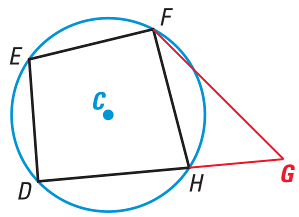
PROVE ▶ $DEFG$ is inscribed in $\odot C$.

Plan for Proof Draw the circle that passes through D, E, and F. Use an *indirect proof* to show that the circle also passes through G. Begin by assuming that G does not lie on $\odot C$.

Case 1 *G lies inside $\odot C$.* Let H be the intersection of $\overrightarrow{DG}$ and $\odot C$. Then $DEFH$ is inscribed in $\odot C$ and $\angle E$ is supplementary to $\angle DHF$ (by proof above). Then $\angle DGF \cong \angle DHF$ by the given information and the Congruent Supplements Theorem. This implies that $\overline{FG} \parallel \overline{FH}$, which is a contradiction.



Case 2 *G lies outside $\odot C$.* Let H be the intersection of $\overrightarrow{DG}$ and $\odot C$. Then $DEFH$ is inscribed in $\odot C$ and $\angle E$ is supplementary to $\angle DHF$ (by proof above). Then $\angle DGF \cong \angle DHF$ by the given information and the Congruent Supplements Theorem. This implies that $\overline{FG} \parallel \overline{FH}$, which is a contradiction.



Because the original assumption leads to a contradiction in both cases, G lies on $\odot C$ and $DEFG$ is inscribed in $\odot C$.