14.6

# Using Sum and Difference Formulas

*What you should learn*

GOAL 1 Evaluate trigonometric functions of the sum or difference of two angles.

GOAL 2 Use sum and difference formulas to solve **real-life** problems, such as determining when pistons in a car engine are at the same height in **Example 6**.

*Why you should learn it*

▼ To model **real-life** quantities, such as the size of an object in an aerial photograph in **Ex. 58**.

REAL LIFE

## GOAL 1 Sum and Difference Formulas

In this lesson you will study formulas that allow you to evaluate trigonometric functions of the sum or difference of two angles.

### SUM AND DIFFERENCE FORMULAS

**SUM FORMULAS**

$\sin(u+v) = \sin u \cos v + \cos u \sin v$

$\cos(u+v) = \cos u \cos v - \sin u \sin v$

$\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

**DIFFERENCE FORMULAS**

$\sin(u-v) = \sin u \cos v - \cos u \sin v$

$\cos(u-v) = \cos u \cos v + \sin u \sin v$

$\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

In general, $\sin(u+v) \neq \sin u + \sin v$. Similar statements can be made for the other trigonometric functions of sums and differences.

### EXAMPLE 1 *Evaluating a Trigonometric Expression*

Find the exact value of (**a**) cos 75° and (**b**) $\tan \frac{\pi}{12}$.

**SOLUTION**

**a.**

$\cos 75° = \cos(45° + 30°)$ — Substitute 45° + 30° for 75°.

$= \cos 45° \cos 30° - \sin 45° \sin 30°$ — Sum formula for cosine

$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$ — Evaluate.

$= \frac{\sqrt{6} - \sqrt{2}}{4}$ — Simplify.

**b.**

$\tan \frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ — Substitute $\frac{\pi}{3} - \frac{\pi}{4}$ for $\frac{\pi}{12}$.

$= \dfrac{\tan \frac{\pi}{3} - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{4}}$ — Difference formula for tangent

$= \dfrac{\sqrt{3} - 1}{1 + (\sqrt{3})(1)}$ — Evaluate.

$= 2 - \sqrt{3}$ — Simplify.

✓**CHECK** Try checking these results with a calculator. For instance, evaluate cos 75° and $\frac{\sqrt{6} - \sqrt{2}}{4}$ to see that both have the same value.

## EXAMPLE 2 *Using a Difference Formula*

Find $\sin(u - v)$ given that $\sin u = -\frac{3}{5}$ with $\pi < u < \frac{3\pi}{2}$ and $\cos v = \frac{12}{13}$ with $0 < v < \frac{\pi}{2}$.

**SOLUTION**

Using a Pythagorean identity and quadrant signs gives $\cos u = -\frac{4}{5}$ and $\sin v = \frac{5}{13}$.

$$\sin(u - v) = \sin u \cos v - \cos u \sin v \qquad \text{Difference formula for sine}$$

$$= -\frac{3}{5}\left(\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) \qquad \text{Substitute.}$$

$$= -\frac{16}{65} \qquad \text{Simplify.}$$

## EXAMPLE 3 *Simplifying an Expression*

Simplify the expression $\cos(x - \pi)$.

**SOLUTION**

$$\cos(x - \pi) = \cos x \cos \pi + \sin x \sin \pi \qquad \text{Difference formula for cosine}$$

$$= (\cos x)(-1) + (\sin x)(0) \qquad \text{Evaluate.}$$

$$= -\cos x \qquad \text{Simplify.}$$

## EXAMPLE 4 *Solving a Trigonometric Equation*

Solve $\sin\left(x + \frac{\pi}{4}\right) + 1 = \sin\left(\frac{\pi}{4} - x\right)$ for $0 \le x < 2\pi$.

**SOLUTION**

$$\sin\left(x + \frac{\pi}{4}\right) + 1 = \sin\left(\frac{\pi}{4} - x\right) \qquad \text{Write original equation.}$$

$$\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} + 1 = \sin \frac{\pi}{4} \cos x - \cos \frac{\pi}{4} \sin x \qquad \text{Use formulas.}$$

$$\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} + 1 = \cos x \sin \frac{\pi}{4} - \sin x \cos \frac{\pi}{4} \qquad \text{Commutative property}$$

$$2 \sin x \cos \frac{\pi}{4} = -1 \qquad \text{Simplify.}$$

$$2(\sin x)\left(\frac{\sqrt{2}}{2}\right) = -1 \qquad \text{Evaluate.}$$

$$\sin x = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \text{Solve for } \sin x.$$

▶ In the interval $0 \le x < 2\pi$, the solutions are $x = \frac{5\pi}{4}$ and $x = \frac{7\pi}{4}$.

✓**CHECK** You can check the solutions with a graphing calculator by graphing each side of the original equation and using the *Intersect* feature to determine the $x$-values for which the expressions are equal.

## GOAL 2 SUM AND DIFFERENCE FORMULAS IN REAL LIFE

REAL LIFE **Biomechanics**

### EXAMPLE 5 *Simplifying a Real-Life Formula*

The force $F$ (in pounds) on a person's back when he or she bends over at an angle $\theta$ is

$$F = \frac{0.6W \sin(\theta + 90°)}{\sin 12°}$$


where $W$ is the person's weight (in pounds).
Simplify this formula.

**SOLUTION**

Begin by expanding $\sin(\theta + 90°)$.

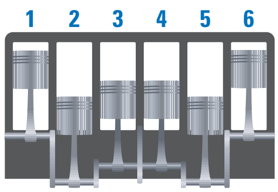
$$F = \frac{0.6W \sin(\theta + 90°)}{\sin 12°} \approx \frac{0.6W(\sin\theta\cos 90° + \cos\theta\sin 90°)}{0.208}$$

$$= \left(\frac{0.6}{0.208}\right)W\,[(\sin\theta)(0) + (\cos\theta)(1)] \approx 2.88W\cos\theta$$

### EXAMPLE 6 *Solving a Trigonometric Equation in Real Life*

**AUTOMOTIVE ENGINEERING** The heights $h$ (in inches) of pistons 1 and 2 in an automobile engine can be modeled by

$$h_1 = 3.75 \sin 733t + 7.5 \quad \text{and} \quad h_2 = 3.75 \sin 733\left(t + \frac{4\pi}{3}\right) + 7.5$$




where $t$ is measured in seconds. How often are these two pistons at the same height?

**SOLUTION**

Let $h_1 = h_2$ and solve for $t$.

$$3.75 \sin 733t + 7.5 = 3.75 \sin 733\left(t + \frac{4\pi}{3}\right) + 7.5$$

$$\sin 733t = \sin 733t \cos\frac{2932\pi}{3} + \cos 733t \sin\frac{2932\pi}{3}$$

$$\sin 733t = (\sin 733t)\left(-\frac{1}{2}\right) + (\cos 733t)\left(-\frac{\sqrt{3}}{2}\right)$$

$$\frac{3}{2}\sin 733t = -\frac{\sqrt{3}}{2}\cos 733t$$

$$\tan 733t = -\frac{\sqrt{3}}{3}$$

$$733t = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) + n\pi$$

$$733t = -\frac{\pi}{6} + n\pi$$

$$t = -\frac{\pi}{4398} + \frac{n\pi}{733}$$

▶ The heights are equal once every $\frac{\pi}{733}$ second. So in one second, the heights are equal the following number of times: $1 \div \frac{\pi}{733} = \frac{733}{\pi} \approx 233.3$.

**FOCUS ON CAREERS**

REAL LIFE **AUTO MECHANIC**
Auto mechanics inspect and repair mechanical and electrical systems of motor vehicles. They may specialize in areas such as transmission systems or diagnostic services.

INTERNET **CAREER LINK**
www.mcdougallittell.com

# GUIDED PRACTICE

**Vocabulary Check ✓**

1. Give the sum and difference formulas for sine, cosine, and tangent.

**Concept Check ✓**

2. Fill in the blanks for each of the following equations.

   a. $\sin(45° - 30°) = \sin \underline{?} \cos 30° - \cos 45° \sin \underline{?}$

   b. $\tan(90° + 60°) = \dfrac{\underline{?} + \tan 60°}{1 - \tan 90° \underline{?}}$

3. Explain how you can evaluate tan 105° using either the sum or difference formula for tangent.

**Skill Check ✓**

**Find the exact value of the expression.**

4. cos 105°
5. sin 15°
6. tan 75°
7. $\cos \frac{11\pi}{12}$
8. $\sin \frac{23\pi}{12}$
9. $\tan \frac{7\pi}{12}$

**Solve the equation for $0 \le x < 2\pi$.**

10. $2 \sin\left(x + \frac{\pi}{3}\right) = \tan \frac{\pi}{3}$
11. $\tan\left(x + \frac{\pi}{6}\right) = \tan\left(x + \frac{\pi}{4}\right)$
12. $\cos\left(x - \frac{\pi}{6}\right) = 1 + \cos\left(x + \frac{\pi}{6}\right)$
13. $\sin\left(x - \frac{4\pi}{3}\right) = 2 \sin\left(x - \frac{\pi}{3}\right)$
14. $4 \sin(x + \pi) = 2 \cos\left(x + \frac{\pi}{2}\right) + 2$
15. $-\cos x = 1 + 2 \cos(x - \pi)$

16. **AUTOMOTIVE ENGINEERING** Look back at Example 6 on page 871. The height $h_3$ of piston 3 in the same engine can be modeled by

$$h_3 = 3.75 \sin 733\left(t + \frac{2\pi}{3}\right) + 7.5$$

where $t$ is measured in seconds. How often is piston 3 the same height as piston 2?

# PRACTICE AND APPLICATIONS

**STUDENT HELP**

**Extra Practice** to help you master skills is on p. 960.

**FINDING VALUES** **Find the exact value of the expression.**

17. cos 210°
18. tan 195°
19. tan 225°
20. sin (−15°)
21. cos (−225°)
22. sin 165°
23. $\tan \frac{11\pi}{12}$
24. $\cos \frac{17\pi}{12}$
25. $\sin\left(-\frac{11\pi}{12}\right)$
26. $\cos \frac{\pi}{12}$
27. $\tan\left(-\frac{5\pi}{12}\right)$
28. $\sin \frac{5\pi}{12}$

**STUDENT HELP**

**HOMEWORK HELP**
**Example 1:** Exs. 17–28
**Example 2:** Exs. 29–40
**Example 3:** Exs. 41–48
**Example 4:** Exs. 49–54
**Examples 5, 6:** Exs. 57–59

**EVALUATING EXPRESSIONS** **Evaluate the expression given $\cos u = \frac{4}{7}$ with $0 < u < \frac{\pi}{2}$ and $\sin v = -\frac{9}{10}$ with $\pi < v < \frac{3\pi}{2}$.**

29. $\sin(u + v)$
30. $\cos(u + v)$
31. $\tan(u + v)$
32. $\sin(u - v)$
33. $\cos(u - v)$
34. $\tan(u - v)$

**STUDENT HELP**

**INTERNET HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 58.

**EVALUATING EXPRESSIONS** **Evaluate the expression given that $\sin u = \frac{3}{5}$ with $\frac{\pi}{2} < u < \pi$ and $\cos v = -\frac{5}{6}$ with $\pi < v < \frac{3\pi}{2}$.**

**35.** $\sin(u + v)$ **36.** $\cos(u + v)$ **37.** $\tan(u + v)$

**38.** $\sin(u - v)$ **39.** $\cos(u - v)$ **40.** $\tan(u - v)$

**SIMPLIFYING EXPRESSIONS** **Simplify the expression.**

**41.** $\tan(x - 2\pi)$ **42.** $\tan(x + \pi)$ **43.** $\sin(x + \pi)$ **44.** $\cos(x + \pi)$

**45.** $\sin\left(x - \frac{\pi}{2}\right)$ **46.** $\cos\left(x + \frac{3\pi}{2}\right)$ **47.** $\cos\left(x + \frac{\pi}{2}\right)$ **48.** $\sin\left(x - \frac{3\pi}{2}\right)$

**SOLVING TRIGONOMETRIC EQUATIONS** **Solve the equation for $0 \le x < 2\pi$.**

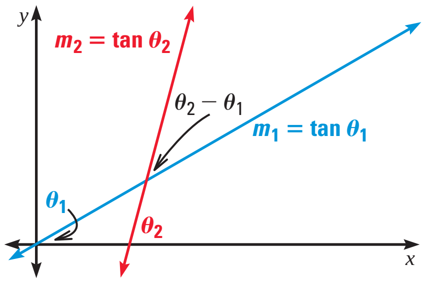
**49.** $\cos\left(x + \frac{\pi}{6}\right) - 1 = \cos\left(x - \frac{\pi}{6}\right)$ **50.** $\sin\left(x + \frac{3\pi}{4}\right) + \sin\left(x - \frac{3\pi}{4}\right) = 1$

**51.** $\sin\left(x + \frac{\pi}{6}\right) + \sin\left(x - \frac{\pi}{6}\right) = 0$ **52.** $\cos\left(x + \frac{\pi}{3}\right) + \cos\left(x - \frac{\pi}{3}\right) = 1$

**53.** $\tan(x + \pi) + 2\sin(x + \pi) = 0$ **54.** $\tan(x + \pi) + \cos\left(x + \frac{\pi}{2}\right) = 0$

**GEOMETRY CONNECTION** **In Exercises 55 and 56, use the following information.** In the figure shown, the acute angle of intersection, $\theta_2 - \theta_1$, of two lines with slopes $m_1$ and $m_2$ is given by:

$$\tan(\theta_2 - \theta_1) = \frac{m_2 - m_1}{1 + m_1 m_2}$$




**55.** Find the acute angle of intersection of the lines $y = \frac{1}{2}x + 3$ and $y = 2x - 3$.

**56.** Find the acute angle of intersection of the lines $y = x + 2$ and $y = 3x + 1$.

**57.** **SOUND WAVES** The pressure $P$ of sound waves on a person's eardrum can be modeled by

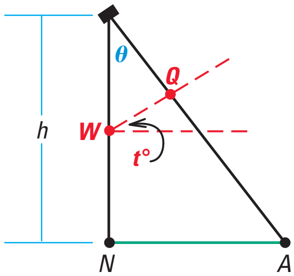
$$P = \frac{a}{r}\cos\left(\frac{2\pi r}{l} - 1100t\right)$$

where $a$ is the maximum sound pressure (in pounds per square foot) at the source, $r$ is the distance (in feet) from the source, $l$ is the length (in feet) of the sound wave, and $t$ is the time (in seconds). Simplify this formula when $r = 16$ feet, $l = 4$ feet, and $a = 0.4$ pound per square foot.

**58.** **AERIAL PHOTOGRAPHY** You are at a height $h$ taking aerial photographs. The ratio of the length $WQ$ of the image to the length $NA$ of the actual object is

$$\frac{WQ}{NA} = \frac{f\tan(\theta - t) + f\tan t}{h\tan\theta}$$

where $f$ is the focal length of the camera, $\theta$ is the angle with the vertical made by the line from the camera to point $A$ and $t$ is the tilt angle of the film. Use the difference formula for tangent to simplify the ratio. Then show that $\frac{WQ}{NA} = \frac{f}{h}$ when $t = 0$.




▶ Source: *Math Applied to Space Science*

**FOCUS ON PEOPLE**

**REAL LIFE** **BARRIE ROKEACH** has more than 20 years of experience as an aerial photographer. He has had more than one dozen individual exhibitions in museums and galleries around the country.

**59.** **FERRIS WHEEL** The heights $h$ (in feet) of two people in different seats on a Ferris wheel can be modeled by

$$h_1 = 28 \cos 10t + 38 \qquad \text{and} \qquad h_2 = 28 \cos 10\left(t - \frac{\pi}{6}\right) + 38$$

where $t$ is the time (in minutes). When are the two people at the same height?

**60.** **CRITICAL THINKING** You can write the sum and difference formulas for cosine as a single equation: $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$. Explain why the symbol $\pm$ is used on the left side, but the symbol $\mp$ is used on the right side. Then use the symbols $\pm$ and $\mp$ to write the sum and difference formulas for sine and tangent as single equations.

**Test Preparation**

**61.** **MULTI-STEP PROBLEM** Suppose two middle-A tuning forks are struck at different times so that their vibrations are slightly out of phase. The combined pressure change $P$ (in pascals) caused by the forks at time $t$ (in seconds) is:

$$P = 3 \sin 880\pi t + 4 \cos 880\pi t$$

**a.** Graph the equation on a graphing calculator using a viewing window of $0 \le x \le 0.5$ and $-6 \le y \le 6$. What do you observe about the graph?

**b.** Write the given model in the form $y = a \cos b(x - h)$.

**c.** Graph the model from part (b) to confirm that the graphs are the same.

**★ Challenge**

**VERIFYING FORMULAS** **Use the difference formula for cosine to verify the following formulas.**

**62.** The sum formula for cosine, by replacing $v$ with $-v$

**63.** The difference formula for tangent, by using the identity $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$

**64.** The difference formula for sine, by using a cofunction identity

EXTRA CHALLENGE
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# MIXED REVIEW

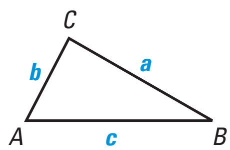
**SIMPLIFYING EXPRESSIONS** **Using the given matrices, simplify the expression.** (Review 4.2)

$$A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ -4 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}, E = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 0 & -2 \end{bmatrix}$$

**65.** $AD + D$ **66.** $3(A + C)$ **67.** $-2AB + B$

**68.** $DE + AC$ **69.** $5CD$ **70.** $AD - CD$

**SOLVING TRIANGLES** **Solve $\triangle ABC$.** (Review 13.5, 13.6)




**71.** $A = 18°, B = 28°, b = 100$ **72.** $A = 60°, C = 95°, c = 5$

**73.** $a = 13, b = 4, c = 11$ **74.** $a = 2, b = 2.5, c = 3$

**SOLVING TRIGONOMETRIC EQUATIONS** **Solve the equation in the interval $0 \le x < 2\pi$.** (Review 14.4 for 14.7)

**75.** $\tan x + \sqrt{3} = 0$ **76.** $4 \cos^2 x - 3 = 0$

**77.** $8 \tan x + 8 = 0$ **78.** $-5 + 8 \cos x = -1$