14.4 Solving Trigonometric Equations

***What** you should learn*

GOAL 1 Solve a trigonometric equation.

GOAL 2 Solve **real-life** trigonometric equations, such as an equation for the number of hours of daylight in Prescott, Arizona, in **Example 6**.

***Why** you should learn it*

▼ To solve many types of **real-life** problems, such as finding the position of the sun at sunrise in **Ex. 58**.

REAL LIFE

GOAL 1 SOLVING A TRIGONOMETRIC EQUATION

In Lesson 14.3 you verified trigonometric identities. In this lesson you will solve trigonometric equations. To see the difference, consider the following equations:

$$\sin^2 x + \cos^2 x = 1 \qquad \textbf{Equation 1}$$

$$\sin x = 1 \qquad \textbf{Equation 2}$$

Equation 1 is an identity because it is true for all real values of x. Equation 2, however, is true only for some values of x. When you find these values, you are solving the equation.

EXAMPLE 1 *Solving a Trigonometric Equation*

Solve $2 \sin x - 1 = 0$.

SOLUTION

First isolate $\sin x$ on one side of the equation.

$2 \sin x - 1 = 0$ **Write original equation.**

$2 \sin x = 1$ **Add 1 to each side.**

$\sin x = \frac{1}{2}$ **Divide each side by 2.**

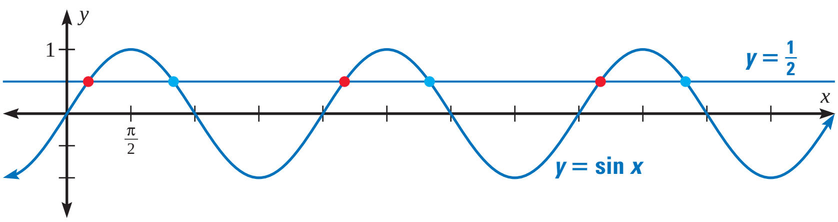
One solution of $\sin x = \frac{1}{2}$ in the interval $0 \le x < 2\pi$ is $x = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$. Another such solution is $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Moreover, because $y = \sin x$ is a periodic function, there are infinitely many other solutions. You can write the general solution as

$$x = \frac{\pi}{6} + 2n\pi \qquad \text{or} \qquad x = \frac{5\pi}{6} + 2n\pi$$

where n is any integer.

✓**CHECK** You can check your answer graphically. Graph $y = \sin x$ and $y = \frac{1}{2}$ in the same coordinate plane and find the points where the graphs intersect. You can see that there are infinitely many such points.



STUDENT HELP

Look Back
For help with inverse trigonometric functions, see p. 792.

EXAMPLE 2 *Solving a Trigonometric Equation in an Interval*

Solve $4\tan^2 x - 1 = 0$ in the interval $0 \le x < 2\pi$.

SOLUTION

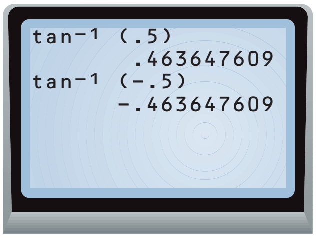
$4\tan^2 x - 1 = 0$ **Write original equation.**

$4\tan^2 x = 1$ **Add 1 to each side.**

$\tan^2 x = \frac{1}{4}$ **Divide each side by 4.**

$\tan x = \pm\frac{1}{2}$ **Take square roots of each side.**

Use a calculator to find values of x for which $\tan x = \pm\frac{1}{2}$, as shown at the right.



The general solution of the equation is

$$x \approx 0.464 + n\pi$$

or

$$x \approx -0.464 + n\pi$$

where n is any integer. The solutions that are in the interval $0 \le x < 2\pi$ are:

$x \approx 0.464$ $\qquad$ $x \approx 0.464 + \pi \approx 3.61$

$x \approx -0.464 + \pi \approx 2.628$ $\qquad$ $x \approx -0.464 + 2\pi \approx 5.82$

✓ **CHECK** Check these solutions by substituting them back into the original equation.

STUDENT HELP

Study Tip
Note that to find the general solution of a trigonometric equation, you must add multiples of the period to the solutions in one cycle.

EXAMPLE 3 *Factoring to Solve a Trigonometric Equation*

Solve $\sin^2 x \cos x = 4\cos x$.

SOLUTION

$\sin^2 x \cos x = 4\cos x$ **Write original equation.**

$\sin^2 x \cos x - 4\cos x = 0$ **Subtract 4 cos *x* from each side.**

$\cos x\,(\sin^2 x - 4) = 0$ **Factor out cos *x*.**

$\cos x\,(\sin x + 2)(\sin x - 2) = 0$ **Factor difference of squares.**

Set each factor equal to 0 and solve for x, if possible.

$\cos x = 0$	$\sin x + 2 = 0$	$\sin x - 2 = 0$
$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$	$\sin x = -2$	$\sin x = 2$

Because neither $\sin x = -2$ nor $\sin x = 2$ has a solution, the only solutions in the interval $0 \le x < 2\pi$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

▶ The general solution is $x = \frac{\pi}{2} + 2n\pi$ or $x = \frac{3\pi}{2} + 2n\pi$ where n is any integer.

STUDENT HELP

Study Tip
Remember not to divide both sides of an equation by a variable expression, such as cos *x*.

EXAMPLE 4 *Using the Quadratic Formula*

Solve $\cos^2 x - 4\cos x + 1 = 0$ in the interval $0 \le x \le \pi$.

SOLUTION

STUDENT HELP

Look Back
For help with the quadratic formula, see p. 291.

Since the equation is in the form $au^2 + bu + c = 0$, you can use the quadratic formula to solve for $u = \cos x$.

$\cos^2 x - 4\cos x + 1 = 0$ **Write original equation.**

$\cos x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$ **Quadratic formula**

$= \dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$ **Simplify.**

≈ 3.73 or 0.268 **Use a calculator.**

$x \approx \cos^{-1} 3.73$ or $x \approx \cos^{-1} 0.268$ **Use inverse cosine.**

No solution $\quad \approx 1.30$ **Use a calculator if possible.**

▸ In the interval $0 \le x \le \pi$, the only solution is $x \approx 1.30$. Check this in the original equation.

. . . .

When solving a trigonometric equation, it is possible to obtain extraneous solutions. Therefore, you should always check your solutions in the original equation.

EXAMPLE 5 *An Equation with Extraneous Solutions*

Solve $1 - \cos x = \sqrt{3}\sin x$ in the interval $0 \le x < 2\pi$.

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

SOLUTION

$1 - \cos x = \sqrt{3}\sin x$ **Write original equation.**

$(1 - \cos x)^2 = (\sqrt{3}\sin x)^2$ **Square both sides.**

$1 - 2\cos x + \cos^2 x = 3\sin^2 x$ **Multiply.**

$1 - 2\cos x + \cos^2 x = 3(1 - \cos^2 x)$ **Pythagorean identity**

$4\cos^2 x - 2\cos x - 2 = 0$ **Quadratic form**

$2\cos^2 x - \cos x - 1 = 0$ **Divide each side by 2.**

$(2\cos x + 1)(\cos x - 1) = 0$ **Factor.**

$2\cos x + 1 = 0$ or $\cos x - 1 = 0$ **Zero product property**

$\cos x = -\dfrac{1}{2} \qquad \cos x = 1$ **Solve for cos *x*.**

$x = \dfrac{2\pi}{3}$ or $x = \dfrac{4\pi}{3} \qquad x = 0$ **Solve for *x*.**

▸ The apparent solution $x = \dfrac{4\pi}{3}$ does not check in the original equation. The only solutions in the interval $0 \le x < 2\pi$ are $x = 0$ and $x = \dfrac{2\pi}{3}$.

FOCUS ON CAREERS

METEOROLOGIST Operational meteorologists, the largest group of specialists in the field, use complex computer models to forecast the weather. Other types of meteorologists include physical meteorologists and climatologists.

CAREER LINK
www.mcdougallittell.com

GOAL 2 SOLVING TRIGONOMETRIC EQUATIONS IN REAL LIFE

EXAMPLE 6 *Solving a Real-Life Trigonometric Equation*

METEOROLOGY The number h of hours of sunlight per day in Prescott, Arizona, can be modeled by

$$h = 2.325 \sin \frac{\pi}{6}(t - 2.667) + 12.155$$

where t is measured in months and $t = 0$ represents January 1. On which days of the year are there 13 hours of sunlight in Prescott? ▶ Source: Gale Research Company

SOLUTION

Method 1 Substitute 13 for h in the model and solve for t.

$$2.325 \sin \frac{\pi}{6}(t - 2.667) + 12.155 = 13$$

$$2.325 \sin \frac{\pi}{6}(t - 2.667) = 0.845$$

$$\sin \frac{\pi}{6}(t - 2.667) \approx 0.363$$

$$\frac{\pi}{6}(t - 2.667) \approx \sin^{-1}(0.363) \quad \text{or} \quad \frac{\pi}{6}(t - 2.667) \approx \pi - \sin^{-1}(0.363)$$

$$\frac{\pi}{6}(t - 2.667) \approx 0.371 \qquad \frac{\pi}{6}(t - 2.667) \approx \pi - 0.371 \approx 2.771$$

$$t - 2.667 \approx 0.709 \qquad t - 2.667 \approx 5.292$$

$$t \approx 3.38 \qquad t \approx 7.96$$

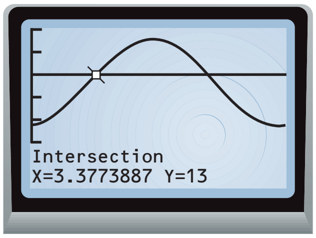
▶ The time $t = 3.38$ represents 3 full months plus $(0.38)(30) \approx 11$ days, or April 11. Likewise, the time $t = 7.96$ represents 7 full months plus $(0.96)(31) \approx 30$ days, or August 30. (Notice that these two days occur about 70 days before and after June 21, which is the date of the summer solstice, the longest day of the year.)

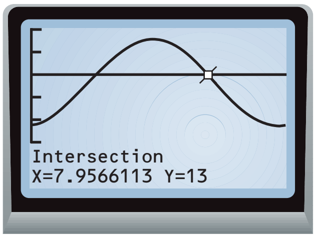
Method 2 Use a graphing calculator. Graph the equations

$$y = 2.325 \sin \frac{\pi}{6}(x - 2.667) + 12.155$$

$$y = 13$$

in the same viewing window. Then use the *Intersect* feature to find the points of intersection.





▶ From the screens above, you can see that $t \approx 3.38$ or $t \approx 7.96$. The time $t = 3.38$ is about April 11, and the time $t = 7.96$ is about August 30.

GUIDED PRACTICE

Vocabulary Check ✓

1. What is the difference between a trigonometric equation and a trigonometric identity?

Concept Check ✓

2. Name several techniques for solving trigonometric equations.

3. **ERROR ANALYSIS** Describe the error(s) in the calculations shown.

$$\cos^2 x = \frac{1}{2}\cos x$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} \text{ or } x = \frac{5\pi}{3} \qquad \text{Solution for } 0 \le x < 2\pi$$

$$x = \frac{\pi}{3} + 2\pi \text{ or } x = \frac{5\pi}{3} + 2\pi \qquad \text{General solution}$$

Skill Check ✓

Solve the equation in the interval $0 \le x < 2\pi$.

4. $2 \cos x + 4 = 5$

5. $3 \sec^2 x - 4 = 0$

6. $\tan^2 x = \cos x \tan^2 x$

7. $5 \cos x - \sqrt{3} = 3 \cos x$

Find the general solution of the equation.

8. $3 \csc x + 5 = 0$

9. $4 \sin x = \sqrt{3}$

10. $1 + \tan^2 x = 6 - 2 \sec^2 x$

11. $2 - 2 \cos^2 x = 5 \sin x + 3$

12. **METEOROLOGY** Look back at the model in Example 6 on page 858. On which days of the year are there 10 hours of sunlight in Prescott, Arizona?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 960.

CHECKING SOLUTIONS **Verify that the given *x*-value is a solution of the equation.**

13. $5 + 4 \cos x - 1 = 0, x = \pi$

14. $\csc x - 2 = 0, x = \frac{5\pi}{6}$

15. $4 \cos^2 x - 3 = 0, x = \frac{\pi}{6}$

16. $3 \tan^3 x - 3 = 0, x = \frac{\pi}{4}$

17. $2 \sin^4 x - \sin^2 x = 0, x = \frac{5\pi}{4}$

18. $2 \cot^4 x - \cot^2 x - 15 = 0, x = \frac{13\pi}{6}$

SOLVING **Find the general solution of the equation.**

19. $2 \cos x - 1 = 0$

20. $3 \tan x - \sqrt{3} = 0$

21. $\sin x = \sin (-x) + 1$

22. $4 \cos x = 2 \cos x + 1$

23. $4 \sin^2 x - 2 = 0$

24. $9 \tan^2 x - 3 = 0$

25. $\sin x \cos x - 2 \cos x = 0$

26. $\sqrt{2} \cos x \sin x - \cos x = 0$

27. $2 \sin^2 x - \sin x = 1$

28. $0 = \cos^2 x - 5 \cos x + 1$

29. $1 - \sin x = \sqrt{3} \cos x$

30. $\sqrt{\sin x} = 2 \sin x - 1$

31. $\cos x - 1 = -\cos x$

32. $6 \sin x = \sin x + 3$

STUDENT HELP

HOMEWORK HELP
Examples 1, 3: Exs. 13–32
Examples 2, 4, 5: Exs. 13–18, 33–56
Example 6: Exs. 57–59

31. $\frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$

SOLVING **Solve the equation in the interval $0 \le x < 2\pi$. Check your solutions.**

33. $5 \cos x - 3 = 0$

34. $3 \sin x = \sin x - 1$

35. $\tan^2 x - 3 = 0$

36. $10 \tan x - 5 = 0$

37. $2 \cos^2 x - \sin x - 1 = 0$

38. $\cos^3 x = \cos x$

39. $\sec^2 x - 2 = 0$

40. $\tan^2 x = \sin x \sec x$

41. $2 \cos x = \sec x$

42. $\cos x \csc^2 x + 3 \cos x = 7 \cos x$

APPROXIMATING SOLUTIONS **Use a graphing calculator to approximate the solutions of the equation in the interval $0 \le x < 2\pi$.**

43. $3 \tan x + 1 = 13$

44. $8 \cos x + 3 = 4$

45. $4 \sin x = -2 \sin x - 5$

46. $3 \sin x + 5 \cos x = 4$

FINDING INTERCEPTS **Find the *x*-intercepts of the graph of the given function in the interval $0 \le x < 2\pi$.**

47. $y = 2 \sin x + 1$

48. $y = 2 \tan^2 x - 6$

49. $y = \sec^2 x - 1$

50. $y = -3 \cos x + \sin x$

FINDING INTERSECTION POINTS **Find the points of intersection of the graphs of the given functions in the interval $0 \le x < 2\pi$.**

51. $y = \sqrt{3} \tan^2 x$
$y = \sqrt{3} - 2 \tan x$

52. $y = 9 \cos^2 x$
$y = \cos^2 x + 8 \cos x - 2$

53. $y = \tan x \sin x$
$y = \cos x$

54. $y = \sin^2 x$
$y = 2 \sin x - 1$

55. $y = 2 - \sin x \tan x$
$y = \cos x$

56. $y = 4 \cos^2 x$
$y = 4 \cos x - 1$

57. **OCEAN TIDES** The *tide*, or depth of the ocean near the shore, changes throughout the day. The depth of the Bay of Fundy can be modeled by

$$d = 35 - 28 \cos \frac{\pi}{6.2} t$$

where d is the water depth in feet and t is the time in hours. Consider a day in which $t = 0$ represents 12:00 A.M. For that day, when do the high and low tides occur? At what time(s) is the water depth $3\frac{1}{2}$ feet?

58. **POSITION OF THE SUN** Cheyenne, Wyoming, has a latitude of 41°N. At this latitude, the position of the sun at sunrise can be modeled by

$$D = 31 \sin \left(\frac{2\pi}{365} t - 1.4\right)$$

where t is the time in days and $t = 1$ represents January 1. In this model, D represents the number of degrees north or south of due east that the sun rises. Use a graphing calculator to determine the days that the sun is more than 20° north of due east at sunrise.

59. **METEOROLOGY** A model for the average daily temperature T (in degrees Fahrenheit) in Kansas City, Missouri, is given by

$$T = 54 + 25.2 \sin \left(\frac{2\pi}{12} t + 4.3\right)$$

where t is measured in months and $t = 0$ represents January 1. What months have average daily temperatures higher than 70°F? Do any months have average daily temperatures below 20°F? ▶ Source: National Climatic Data Center

FOCUS ON APPLICATIONS

REAL LIFE **THE BAY OF FUNDY** is the portion of the Atlantic Ocean that runs along the southern coast of New Brunswick in Canada. It is the site of the highest tides in the world.

INTERNET **APPLICATION LINK**
www.mcdougallittell.com

Test Preparation

60. MULTIPLE CHOICE What is the general solution of the equation $\cos x + \sqrt{2} = -\cos x$? Assume n is an integer.

Ⓐ $x = \frac{\pi}{4} + 2n\pi$ Ⓑ $x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{7\pi}{4} + 2n\pi$

Ⓒ $x = \frac{3\pi}{4} + 2n\pi$ Ⓓ $x = \frac{3\pi}{4} + 2n\pi$ or $x = \frac{5\pi}{4} + 2n\pi$

Ⓔ $x = \frac{3\pi}{4} + n\pi$ or $x = \frac{5\pi}{4} + n\pi$

61. MULTIPLE CHOICE Find the points of intersection of the graphs of $y = 2 + \sin x$ and $y = 3 - \sin x$ in the interval $0 \le x < 2\pi$.

Ⓐ $\left(\frac{\pi}{6}, \frac{5}{2}\right), \left(\frac{5\pi}{6}, \frac{5}{2}\right)$ Ⓑ $\left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right)$ Ⓒ $\left(\frac{\pi}{3}, \frac{5}{2}\right), \left(\frac{4\pi}{3}, \frac{5}{2}\right)$

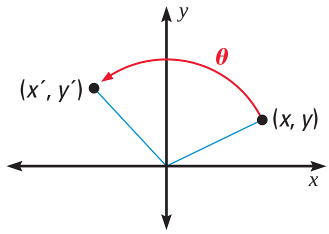
Ⓓ $\left(\frac{\pi}{3}, \frac{1}{2}\right), \left(\frac{4\pi}{3}, \frac{1}{2}\right)$ Ⓔ $\left(\frac{\pi}{6}, \frac{5}{2}\right), \left(\frac{11\pi}{6}, \frac{5}{2}\right)$

★ **Challenge**

MATRICES In Exercises 62 and 63, use the following information.
Matrix multiplication can be used to rotate a point (x, y) counter clockwise about the origin through an angle θ. The coordinates of the resulting point (x', y') are determined by the following matrix equation:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

62. The point (4, 1) is rotated counter clockwise about the origin through an angle of $\frac{\pi}{3}$. What are the coordinates of the resulting point?



63. Through what angle θ must the point (2, 4) be rotated to produce $(x', y') = \left(-2 + \sqrt{3}, 1 + 2\sqrt{3}\right)$?

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

USING COMPLEMENTS Two six-sided dice are rolled. Find the probability of the given event. (Review 12.4)

64. The sum is less than or equal to 10.

65. The sum is not 4.

66. The sum is not 2 or 12.

67. The sum is greater than 3.

GRAPHING Graph the function. (Review 14.1, 14.2 for 14.5)

68. $y = \sin 3x$

69. $y = 2\cos 4x$

70. $y = 10\sin 2x$

71. $y = 3\cos\frac{1}{2}x$

72. $y = \frac{1}{4}\tan x$

73. $y = 3\tan\frac{1}{2}x - 2$

74. $y = \cos\frac{1}{2}x + \pi$

75. $y = 3 + \tan\left(x - \frac{3\pi}{2}\right)$

76. $y = -\sin 3\pi(x + 4) + 1$

77. SURVEYING Suppose you are trying to determine the width w of a small pond. You stand at a point 43 feet from one end of the pond and 50 feet from the other end. The angle formed by your lines of sight to each end of the pond measures 45°. How wide is the pond? (Review 13.6)