14.3 Verifying Trigonometric Identities

What you should learn

GOAL 1 Use trigonometric identities to simplify trigonometric expressions and to verify other identities.

GOAL 2 Use trigonometric identities to solve **real-life** problems, such as comparing the speeds at which people pedal exercise machines in **Example 7**.

Why you should learn it

▼ To simplify **real-life** trigonometric expressions, such as the parametric equations that describe a carousel's motion in **Ex. 65**.

REAL LIFE

GOAL 1 USING TRIGONOMETRIC IDENTITIES

In this lesson you will use *trigonometric identities* to evaluate trigonometric functions, simplify trigonometric expressions, and verify other identities.

ACTIVITY Developing Concepts

Investigating Trigonometric Identities

Use a graphing calculator to graph each side of the equation in the same viewing window. What do you notice about the graphs? Is the equation true for (**a**) no x-values, (**b**) some x-values, or (**c**) all x-values? (Set your calculator in radian mode and use $-2\pi \le x \le 2\pi$ and $-2 \le y \le 2$.)

1. $\sin^2 x + \cos^2 x = 1$ **2.** $\sin(-x) = -\sin x$

3. $\sin x = -\cos x$ **4.** $\cos x = 1.5$

In the activity you may have discovered that some trigonometric equations are true for all values of x (in their domain). Such equations are called **trigonometric identities**. In Lesson 13.1 you used reciprocal identities to find the values of the cosecant, secant, and cotangent functions. These and other fundamental identities are listed below.

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

RECIPROCAL IDENTITIES

$\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$

TANGENT AND COTANGENT IDENTITIES

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

PYTHAGOREAN IDENTITIES

$\sin^2\theta + \cos^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$

COFUNCTION IDENTITIES

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

NEGATIVE ANGLE IDENTITIES

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$

EXAMPLE 1 Finding Trigonometric Values

Given that $\sin \theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find the values of the other five trigonometric functions of θ.

SOLUTION

Begin by finding $\cos \theta$.

$\sin^2 \theta + \cos^2 \theta = 1$	Write Pythagorean identity.
$\left(\frac{3}{5}\right)^2 + \cos^2 \theta = 1$	Substitute $\frac{3}{5}$ for $\sin \theta$.
$\cos^2 \theta = 1 - \left(\frac{3}{5}\right)^2$	Subtract $\left(\frac{3}{5}\right)^2$ from each side.
$\cos^2 \theta = \frac{16}{25}$	Simplify.
$\cos \theta = \pm\frac{4}{5}$	Take square roots of each side.
$\cos \theta = -\frac{4}{5}$	Because θ is in Quadrant II, $\cos \theta$ is negative.

Now, knowing $\sin \theta$ and $\cos \theta$, you can find the values of the other four trigonometric functions.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3} \qquad \sec \theta = \frac{1}{\cos \theta} = \frac{1}{-\frac{4}{5}} = -\frac{5}{4}$$

EXAMPLE 2 Simplifying a Trigonometric Expression

Simplify the expression $\sec \theta \tan^2 \theta + \sec \theta$.

SOLUTION

$\sec \theta \tan^2 \theta + \sec \theta = \sec \theta (\sec^2 \theta - 1) + \sec \theta$	Pythagorean identity
$= \sec^3 \theta - \sec \theta + \sec \theta$	Distributive property
$= \sec^3 \theta$	Simplify.

EXAMPLE 3 Simplifying a Trigonometric Expression

Simplify the expression $\cos\left(\frac{\pi}{2} - x\right) \cot x$.

SOLUTION

$\cos\left(\frac{\pi}{2} - x\right) \cot x = \sin x \cot x$	Cofunction identity
$= \sin x \left(\frac{\cos x}{\sin x}\right)$	Cotangent identity
$= \cos x$	Simplify.

You can use the fundamental identities on page 848 to *verify* new trigonometric identities. A *verification* of an identity is a chain of equivalent expressions showing that one side of the identity is equal to the other side. When verifying an identity, begin with the expression from one side and manipulate it algebraically until it is identical to the other side.

EXAMPLE 4 *Verifying a Trigonometric Identity*

Verify the identity $\cot(-\theta) = -\cot\theta$.

SOLUTION

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} \quad \text{Cotangent identity}$$

$$= \frac{\cos\theta}{-\sin\theta} \quad \text{Negative angle identities}$$

$$= -\cot\theta \quad \text{Cotangent identity}$$

STUDENT HELP

Study Tip

Verifying an identity is *not* the same as solving an equation. When verifying an identity you should *not* use any properties of equality, such as adding the same number or expression to both sides.

EXAMPLE 5 *Verifying a Trigonometric Identity*

Verify the identity $\frac{\cot^2 x}{\csc x} = \csc x - \sin x$.

SOLUTION

$$\frac{\cot^2 x}{\csc x} = \frac{\csc^2 x - 1}{\csc x} \quad \text{Pythagorean identity}$$

$$= \frac{\csc^2 x}{\csc x} - \frac{1}{\csc x} \quad \text{Write as separate fractions.}$$

$$= \csc x - \frac{1}{\csc x} \quad \text{Simplify.}$$

$$= \csc x - \sin x \quad \text{Reciprocal identity}$$

EXAMPLE 6 *Verifying a Trigonometric Identity*

Verify the identity $\frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$.

SOLUTION

$$\frac{\sin x}{1 - \cos x} = \frac{\sin x\,(1 + \cos x)}{(1 - \cos x)(1 + \cos x)} \quad \text{Multiply by } \frac{1 + \cos x}{1 + \cos x}.$$

$$= \frac{\sin x\,(1 + \cos x)}{1 - \cos^2 x} \quad \text{Simplify denominator.}$$

$$= \frac{\sin x\,(1 + \cos x)}{\sin^2 x} \quad \text{Pythagorean identity}$$

$$= \frac{1 + \cos x}{\sin x} \quad \text{Simplify.}$$

STUDENT HELP

Study Tip

In Example 6, notice how multiplying by an expression equal to 1 allows you to write an expression in an equivalent form.

GOAL 2 USING TRIGONOMETRIC IDENTITIES IN REAL LIFE

In Lesson 13.7 you learned that parametric equations can be used to describe linear and parabolic motion. They can be used to describe other types of motion as well.

EXAMPLE 7 *Using Parametric Equations in Real Life*

PHYSICAL FITNESS You and Sara are riding exercise machines that involve pedaling. The following parametric equations describe the motion of your feet and Sara's feet:

YOU:	SARA:
$x = 8 \cos 4\pi t$	$x = 10 \cos 2\pi t$
$y = 8 \sin 4\pi t$	$y = 6 \sin 2\pi t$

In each case, x and y are measured in inches and t is measured in seconds.

a. Describe the paths followed by your feet and Sara's feet.

b. Who is pedaling faster (in revolutions per second)?

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

SOLUTION

a. Use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ to eliminate the parameter t.

YOU:	SARA:	
$\frac{x}{8} = \cos 4\pi t$	$\frac{x}{10} = \cos 2\pi t$	Isolate the cosine.
$\frac{y}{8} = \sin 4\pi t$	$\frac{y}{6} = \sin 2\pi t$	Isolate the sine.
$\cos^2 4\pi t + \sin^2 4\pi t = 1$	$\cos^2 2\pi t + \sin^2 2\pi t = 1$	Pythagorean identity
$\left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2 = 1$	$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$	Substitute.
$x^2 + y^2 = 64$	$\frac{x^2}{100} + \frac{y^2}{36} = 1$	Simplify.

▶ Your feet follow a circle with a radius of 8 inches. Sara's feet follow an ellipse whose major axis is 20 inches long and whose minor axis is 12 inches long.

b. The number of revolutions per second for you and Sara is the reciprocal of the common period of the corresponding parametric functions.

YOU:	SARA:
$\frac{1}{\frac{2\pi}{4\pi}} = \frac{1}{\frac{1}{2}} = 2$	$\frac{1}{\frac{2\pi}{2\pi}} = \frac{1}{1} = 1$

▶ In one second, your feet travel around 2 times and Sara's feet travel around 1 time. So, you are pedaling faster.

✓**CHECK** To check your results, set a graphing calculator to parametric, radian, and simultaneous modes. Enter both sets of parametric equations with $0 \le t \le 1$ and a t-step of 0.01. As the paths are graphed, you can see that your path is traced faster.

FOCUS ON APPLICATIONS

REAL LIFE **ELLIPTICAL TRAINERS** provide excellent aerobic exercise by combining both lower and upper body movements. The smooth elliptical motion produces less impact than experienced on a treadmill.

GUIDED PRACTICE

Vocabulary Check ✓

1. What is a trigonometric identity?

Concept Check ✓

2. Is sec $(-\theta)$ equal to sec θ or $-$sec θ? How do you know?

3. Verify the identity $1 - \sin^2 x \cot^2 x = \sin^2 x$. Is there more than one way to verify the identity? If so, tell which way you think is easier and why.

4. **ERROR ANALYSIS** Describe what is wrong with the simplification shown.

$$\begin{aligned} \cos x - \cos x \sin^2 x &= \cos x - \cos x\,(1 + \cos^2 x) \\ &= \cos x - \cos x - \cos^3 x \\ &= -\cos^3 x \end{aligned}$$

Skill Check ✓

Find the values of the other five trigonometric functions of θ.

5. $\cos\theta = -\frac{3}{5}, \frac{\pi}{2} < \theta < \pi$

6. $\tan\theta = \frac{2}{3}, 0 < \theta < \frac{\pi}{2}$

7. $\sec\theta = \frac{4}{3}, \frac{3\pi}{2} < \theta < 2\pi$

8. $\sin\theta = -\frac{1}{2}, \pi < \theta < \frac{3\pi}{2}$

Simplify the expression.

9. $\dfrac{(\sec x + 1)(\sec x - 1)}{\tan x}$

10. $\sin\left(\frac{\pi}{2} - x\right) \sec x$

11. $\cos^2\left(\frac{\pi}{2} - x\right) + \cos^2(-x)$

Verify the identity.

12. $\dfrac{1}{\sin(-x)} = -\csc x$

13. $\cot x \tan(-x) = -1$

14. $\csc x \tan x = \sec x$

15. **PHYSICAL FITNESS** Look back at Example 7 on page 851. Suppose your friend Pete starts riding another machine that involves pedaling. The motion of his feet is described by the equations $x = 10\cos\frac{5\pi}{2}t$ and $y = 6\sin\frac{5\pi}{2}t$. What type of path are his feet following?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 959.

FINDING VALUES **Find the values of the other five trigonometric functions of θ.**

16. $\cos\theta = \frac{1}{\sqrt{5}}, 0 < \theta < \frac{\pi}{2}$

17. $\tan\theta = \frac{3}{8}, 0 < \theta < \frac{\pi}{2}$

18. $\sin\theta = \frac{5}{6}, 0 < \theta < \frac{\pi}{2}$

19. $\sin\theta = \frac{3}{5}, \frac{\pi}{2} < \theta < \pi$

20. $\cot\theta = -\frac{9}{4}, \frac{3\pi}{2} < \theta < 2\pi$

21. $\cos\theta = -\frac{11}{12}, \frac{\pi}{2} < \theta < \pi$

22. $\csc\theta = \frac{7}{5}, 0 < \theta < \frac{\pi}{2}$

23. $\sec\theta = -\frac{10}{3}, \pi < \theta < \frac{3\pi}{2}$

24. $\tan\theta = -\frac{1}{6}, \frac{\pi}{2} < \theta < \pi$

25. $\sec\theta = 2, \frac{3\pi}{2} < \theta < 2\pi$

26. $\csc\theta = -\frac{5}{3}, \pi < \theta < \frac{3\pi}{2}$

27. $\cot\theta = -\sqrt{3}, \frac{3\pi}{2} < \theta < 2\pi$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 16–27
Examples 2, 3: Exs. 28–43
Examples 4–6: Exs. 44–53
Example 7: Exs. 61–64

SIMPLIFYING EXPRESSIONS **Simplify the expression.**

28. $\cot x \sec x$

29. $\dfrac{\cos(-x)}{\sin(-x)}$

30. $\sec x \cos(-x) - \sin^2 x$

31. $\sin x\,(1 + \cot^2 x)$

32. $1 - \sin^2\left(\frac{\pi}{2} - x\right)$

33. $\dfrac{\tan\left(\frac{\pi}{2} - x\right)}{\csc x}$

34. $\cos\left(\frac{\pi}{2} - x\right)\csc x$

35. $\dfrac{\sin(-x)}{\csc x} + \cos^2(-x)$

36. $\dfrac{\cos^2 x \tan^2(-x) - 1}{\cos^2 x}$

37. $\sec^2 x - \tan^2 x$

38. $\dfrac{\tan\left(\frac{\pi}{2} - x\right)\sec x}{1 - \csc^2 x}$

39. $\dfrac{\cos\left(\frac{\pi}{2} - x\right) - 1}{1 + \sin(-x)}$

40. $\dfrac{\cot x \cos x}{\tan(-x)\sin\left(\frac{\pi}{2} - x\right)}$

41. $\dfrac{\sec x \sin x + \cos\left(\frac{\pi}{2} - x\right)}{1 + \sec x}$

42. $\cot^2 x + \sin^2 x + \cos^2(-x)$

43. $\tan\left(\frac{\pi}{2} - x\right)\cot x - \csc^2 x$

VERIFYING IDENTITIES **Verify the identity.**

44. $\cos x \sec x = 1$

45. $\tan x \csc x \cos x = 1$

46. $\cos\left(\frac{\pi}{2} - x\right)\cot x = \cos x$

47. $2 - \sec^2 x = 1 - \tan^2 x$

48. $\sin x + \cos x \cot x = \csc x$

49. $\dfrac{\cos^2 x + \sin^2 x}{1 + \tan^2 x} = \cos^2 x$

50. $\dfrac{\sin^2(-x)}{\tan^2 x} = \cos^2 x$

51. $\dfrac{\sin\left(\frac{\pi}{2} - x\right) - 1}{1 - \cos(-x)} = -1$

52. $\dfrac{1 + \sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} = 2\sec x$

53. $\dfrac{\cos(-x)}{1 + \sin(-x)} = \sec x + \tan x$

IDENTIFYING CONICS **Use a graphing calculator set in parametric mode to graph the parametric equations. Use a trigonometric identity to determine whether the graph is *a circle, an ellipse,* or *a hyperbola*. (Use a square viewing window.)**

54. $x = 6\cos t,\ y = 6\sin t$

55. $x = 5\sec t,\ y = \tan t$

56. $x = 2\cos t,\ y = 3\sin t$

57. $x = 8\cos \pi t,\ y = 8\sin \pi t$

58. $x = 2\cot 2t,\ y = 3\csc 2t$

59. $x = \cos\frac{t}{2},\ y = 4\sin\frac{t}{2}$

60. CRITICAL THINKING A function f is *odd* if $f(-x) = -f(x)$. A function f is *even* if $f(-x) = f(x)$. Which of the six trigonometric functions are odd? Which of them are even?

61. SHADOW OF A SUNDIAL The length s of a shadow cast by a vertical *gnomon* (column or shaft on a sundial) of height h when the angle of the sun above the horizon is θ can be modeled by this equation:

$$s = \frac{h\sin(90^\circ - \theta)}{\sin\theta}$$

This equation was developed by Abu Abdullah al-Battani (circa A.D. 920). Show that the equation is equivalent to $s = h\cot\theta$. ▶ Source: *Trigonometric Delights*

GLEASTON WATER MILL **In Exercises 62–64, use the following information.**
Suppose you have constructed a working scale model of the water wheel at the Gleaston Water Mill. The parametric equations that describe the motion of one of the paddles on each of the water wheels are as follows.

Actual waterwheel:	**Scale model:**
$x = 9 \cos 8\pi t$	$x = 0.5 \cos 15\pi t$
$y = 9 \sin 8\pi t$	$y = 0.5 \sin 15\pi t$

In each case, x and y are measured in feet and t is measured in minutes.

62. Use a graphing calculator to graph both sets of parametric equations.

63. How is your scale model different from the actual waterwheel?

64. How many revolutions does each wheel make in 5 minutes?

Test Preparation

65. MULTI-STEP PROBLEM The Dentzel Carousel in Glen Echo Park near Washington, D.C., is one of about 135 functioning antique carousels in the United States. The platform of the carousel is about 48 feet in diameter and makes about 5 revolutions per minute. ▶ Source: National Park Service

a. Find parametric equations that describe the ride's motion.

b. Suppose the platform of the carousel had a diameter of 38 feet and made about 4.5 revolutions per minute. Find parametric equations that would describe the ride's motion.

2

c. *Writing* How are the parametric equations affected when the speed is changed? How are the equations affected when the diameter is changed?

★ Challenge

66. Use the definitions of sine and cosine from Lesson 13.3 to derive the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$.

67. Use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ to derive the other Pythagorean identities, $1 + \tan^2 \theta = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$.

MIXED REVIEW

QUADRATIC EQUATIONS **Solve the equation by factoring.** (Review 5.2 for 14.4)

68. $x^2 - 5x - 14 = 0$ **69.** $x^2 + 5x - 36 = 0$ **70.** $x^2 - 19x + 88 = 0$

71. $2x^2 - 7x - 15 = 0$ **72.** $36x^2 - 16 = 0$ **73.** $9x^2 - 1 = 0$

EVALUATING EXPRESSIONS **Evaluate the expression without using a calculator. Give your answer in both radians and degrees.** (Review 13.4 for 14.4)

74. $\cos^{-1} \frac{\sqrt{2}}{2}$ **75.** $\tan^{-1} \sqrt{3}$ **76.** $\sin^{-1} \left(-\frac{\sqrt{3}}{2}\right)$

77. $\sin^{-1} \frac{1}{2}$ **78.** $\tan^{-1} (-1)$ **79.** $\cos^{-1} \left(-\frac{1}{2}\right)$

GRAPHING **Draw one cycle of the function's graph.** (Review 14.1)

80. $y = 4 \sin x$ **81.** $y = 2 \cos x$ **82.** $y = \tan 4\pi x$

83. $y = 3 \tan \pi x$ **84.** $y = 5 \cos 2x$ **85.** $y = 10 \sin 4x$