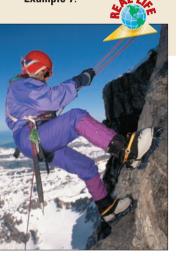
What you should learn

GOAL Graph translations and reflections of sine and cosine graphs.

GOAL 2 Graph translations and reflections of tangent graphs, as applied in Ex. 61.

Why you should learn it

▼ To model real-life quantities, such as the height above the ground of a person rappelling down a cliff in Example 7.



Translations and Kerre Trigonometric Graphs **Translations and Reflections of**

GOAL 1

GRAPHING SINE AND COSINE FUNCTIONS

In previous chapters you learned that the graph of $y = a \cdot f(x - h) + k$ is related to the graph of $y = |a| \cdot f(x)$ by horizontal and vertical translations and by a reflection when a is negative. This also applies to sine, cosine, and tangent functions.

TRANSFORMATIONS OF SINE AND COSINE GRAPHS

To obtain the graph of

$$y = a \sin b(x - h) + k$$
 or $y = a \cos b(x - h) + k$

transform the graph of $y = |a| \sin bx$ or $y = |a| \cos bx$ as follows.

VERTICAL SHIFT Shift the graph k units vertically.

HORIZONTAL SHIFT Shift the graph *h* units horizontally.

REFLECTION If a < 0, reflect the graph in the line y = k after any vertical and horizontal shifts have been performed.

EXAMPLE 1 Graphing a Vertical Translation

Graph $y = -2 + 3 \sin 4x$.

SOLUTION

Because the graph is a transformation of the graph of $y = 3 \sin 4x$, the amplitude is 3 and the period is $\frac{2\pi}{4} = \frac{\pi}{2}$. By comparing the given equation to the general equation $y = a \sin b(x - h) + k$, you can see that h = 0, k = -2, and a > 0. Therefore, translate the graph of $y = 3 \sin 4x$ down 2 units.

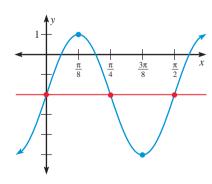
The graph oscillates 3 units up and down from its center line y = -2. Therefore, the maximum value of the function is -2 + 3 = 1 and the minimum value of the function is -2 - 3 = -5.

The five key points are:

On
$$y = k$$
: $(0, -2)$; $(\frac{\pi}{4}, -2)$; $(\frac{\pi}{2}, -2)$

Maximum: $\left(\frac{\pi}{8}, 1\right)$

Minimum: $\left(\frac{3\pi}{8}, -5\right)$



✓ CHECK You can check your graph with a graphing calculator. Use the *Maximum*, Minimum, and Intersect features to check the key points.

Graph
$$y = 2\cos\frac{2}{3}\left(x - \frac{\pi}{4}\right)$$
.

SOLUTION

Because the graph is a transformation of the graph of $y = 2\cos\frac{2}{3}x$, the amplitude is 2 and the period is $\frac{2\pi}{2} = 3\pi$. By comparing the given equation to the general equation $y = a \cos b(x - h) + k$, you can see that $h = \frac{\pi}{4}$, k = 0, and a > 0. Therefore, translate the graph of $y = 2 \cos \frac{2}{3}x$ right $\frac{\pi}{4}$ unit. (Notice that the maximum occurs

The five key points are:

 $\frac{\pi}{4}$ unit to the right of the y-axis.)

STUDENT HELP

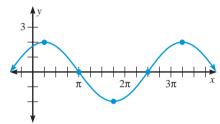
Study Tip

When graphing translations of functions, you may find it helpful to graph the basic function first and then translate the graph.

On
$$y = k$$
: $\left(\left(\frac{1}{4} \cdot 3\pi\right) + \frac{\pi}{4}, 0\right) = (\pi, 0);$
 $\left(\left(\frac{3}{4} \cdot 3\pi\right) + \frac{\pi}{4}, 0\right) = \left(\frac{5\pi}{2}, 0\right)$
Maximums: $\left(0 + \frac{\pi}{4}, 2\right) = \left(\frac{\pi}{4}, 2\right);$
 $\left(3\pi + \frac{\pi}{4}, 2\right) = \left(\frac{13\pi}{4}, 2\right)$

Maximums:
$$\left(0 + \frac{\pi}{4}, 2\right) = \left(\frac{\pi}{4}, 2\right);$$
 $\left(3\pi + \frac{\pi}{4}, 2\right) = \left(\frac{13\pi}{4}, 2\right)$

Minimum:
$$\left(\left(\frac{1}{2} \cdot 3\pi\right) + \frac{\pi}{4}, -2\right) = \left(\frac{7\pi}{4}, -2\right)$$



EXAMPLE 3 Graphing a Reflection

Graph $y = -3 \sin x$.

SOLUTION

Because the graph is a reflection of the graph of $y = 3 \sin x$, the amplitude is 3 and the period is 2π . When you plot the five key points on the graph, note that the intercepts are the same as they are for the graph of $y = 3 \sin x$. However, when the graph is reflected in the x-axis, the maximum becomes a minimum and the minimum becomes a maximum.

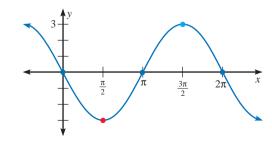
The five key points are:

On
$$y = k$$
: $(0, 0)$; $(2\pi, 0)$;

$$\left(\frac{1}{2} \cdot 2\pi, 0\right) = (\pi, 0)$$

$$\mathbf{Minimum: } \left(\frac{1}{4} \cdot 2\pi, -3\right) = \left(\frac{\pi}{2}, -3\right)$$

$$\mathbf{Maximum:} \ \left(\frac{3}{4} \cdot 2\pi, 3\right) = \left(\frac{3\pi}{2}, 3\right)$$



The next example shows how to graph a function when multiple transformations are involved.

EXAMPLE 4 Combining a Translation and a Reflection

Graph
$$y = -\frac{1}{2}\cos(2x + 3\pi) + 1$$
.

line y = 1. The five key points are:

SOLUTION

Begin by rewriting the function in the form $y = a \cos b(x - h) + k$:

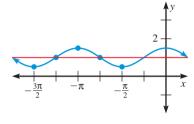
$$y = -\frac{1}{2}\cos(2x + 3\pi) + 1 = -\frac{1}{2}\cos 2\left[x - \left(-\frac{3\pi}{2}\right)\right] + 1$$

The amplitude is $\frac{1}{2}$ and the period is $\frac{2\pi}{2} = \pi$. Since $h = -\frac{3\pi}{2}$, k = 1, and a < 0, the graph of $y = \frac{1}{2}\cos 2x$ is shifted **left** $\frac{3\pi}{2}$ units and up 1 unit, and then reflected in the

On y = k: $\left(\left(\frac{1}{4} \cdot \pi\right) - \frac{3\pi}{2}, 1\right) = \left(-\frac{5\pi}{4}, 1\right)$; $\left(\left(\frac{3}{4} \cdot \pi\right) - \frac{3\pi}{2}, 1\right) = \left(-\frac{3\pi}{4}, 1\right)$

Minimums: $\left(0 - \frac{3\pi}{2}, 1 - \frac{1}{2}\right) = \left(-\frac{3\pi}{2}, \frac{1}{2}\right);$ $\left(\pi - \frac{3\pi}{2}, 1 - \frac{1}{2}\right) = \left(-\frac{\pi}{2}, \frac{1}{2}\right)$

Maximum: $\left(\left(\frac{1}{2} \cdot \pi\right) - \frac{3\pi}{2}, 1 + \frac{1}{2}\right) = \left(-\pi, \frac{3}{2}\right)$



EXAMPLE 5

Modeling Circular Motion

FERRIS WHEEL You are riding a Ferris wheel. Your height h (in feet) above the ground at any time t (in seconds) can be modeled by the following equation:

$$h = 25 \sin \frac{\pi}{15} \left(t - 7.5 \right) + 30$$

The Ferris wheel turns for 135 seconds before it stops to let the first passengers off.

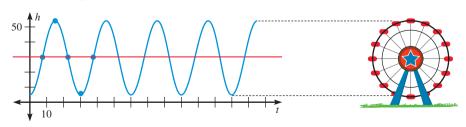
- **a.** Graph your height above the ground as a function of time.
- **b.** What are your minimum and maximum heights above the ground?

SOLUTION

a. The amplitude is 25 and the period is $\frac{2\pi}{\frac{\pi}{15}} = 30$. The wheel turns $\frac{135}{30} = 4.5$ times

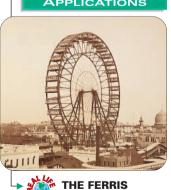
in 135 seconds, so the graph shows 4.5 cycles.

The five key points are (7.5, 30), (15, 55), (22.5, 30), (30, 5), and (37.5, 30).



b. Since the amplitude is 25 and the graph is shifted up 30 units, the maximum height is 30 + 25 = 55 feet and the minimum height is 30 - 25 = 5 feet.

Focus on APPLICATIONS



wheel was invented in 1893 by George Washington Gale Ferris for the World's Columbian Exposition in Chicago. Each car held 60 people.



Graphing tangent functions using translations and reflections is similar to graphing sine and cosine functions.

TRANSFORMATIONS OF TANGENT GRAPHS

To obtain the graph of $y = a \tan b(x - h) + k$, transform the graph of $y = |a| \tan bx$ as follows.

- Shift the graph k units vertically and h units horizontally.
- Then, if a < 0, reflect the graph in the line y = k.

EXAMPLE 6 Combining a Translation and a Reflection

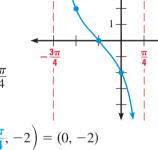
Graph
$$y = -2 \tan \left(x + \frac{\pi}{4} \right)$$
.

SOLUTION

The graph is a transformation of the graph of $y = 2 \tan x$, so the period is π . By comparing the given equation to $y = a \tan b(x - h) + k$, you can see that $h = -\frac{\pi}{4}$, k = 0, and a < 0. Therefore, translate the graph of $y = 2 \tan x$ left $\frac{\pi}{4}$ unit and then reflect it in the x-axis.

Asymptotes:
$$x = -\frac{\pi}{2 \cdot 1} - \frac{\pi}{4} = -\frac{3\pi}{4}$$
; $x = \frac{\pi}{2 \cdot 1} - \frac{\pi}{4} = \frac{\pi}{4}$
On $y = k$: $(h, k) = \left(-\frac{\pi}{4}, 0\right)$

Halfway points:
$$\left(-\frac{\pi}{4 \cdot 1} - \frac{\pi}{4}, 2\right) = \left(-\frac{\pi}{2}, 2\right); \left(\frac{\pi}{4 \cdot 1} - \frac{\pi}{4}, -2\right) = (0, -2)$$





EXAMPLE 7

Modeling with a Tangent Function

You are standing 200 feet from the base of a 180 foot cliff. Your friend is rappelling down the cliff. Write and graph a model for your friend's distance d from the top as a function of her angle of elevation θ .

SOLUTION

Use the tangent function to write an equation relating d and θ .

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{180 - d}{200}$$

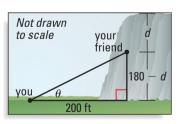
Definition of tangent

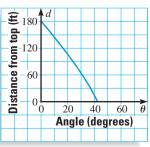
$$200 \tan \theta = 180 - d$$

Multiply each side by 200.

$$d = -200 \tan \theta + 180$$

Solve for d.





GUIDED PRACTICE

Concept Check

- **1.** Complete this statement: A(n) ? shifts a graph horizontally or vertically.
- **2.** How is the graph of $y = -2 \cos 3x$ related to the graph of $y = 2 \cos 3x$?
- **3.** How is the graph of $y = \tan 2(x \pi)$ related to the graph of $y = \tan 2x$?

Skill Check

State whether the graph of the function is a vertical shift, a horizontal shift, and/or a reflection of the graph of $y = 4 \cos 2x$.

4.
$$y = 3 + 4 \cos 2x$$

5.
$$y = 4 \cos (2x + 1)$$
 6. $y = -4 \cos 2x$

6.
$$v = -4 \cos 2x$$

7.
$$y = 4 \cos 2(x + 1)$$

7.
$$y = 4\cos 2(x+1)$$
 8. $y = 4\cos (2x-1) + 3$ **9.** $y = -3 - 4\cos 2x$

9.
$$y = -3 - 4 \cos 2x$$

Graph the function.

10.
$$y = 3 \sin(x + \pi)$$

11.
$$y = -2 \cos x + \frac{1}{2}$$

11.
$$y = -2 \cos x + 1$$
 12. $y = 2 \tan (x - \pi)$

13.
$$y = -\sin \pi(x-2) + 3$$
 14. $y = 4\cos 2(x-\pi) + 1$ **15.** $y = 5 - \tan 2(x-\pi)$

14.
$$y = 4 \cos 2(x - \pi) +$$

15.
$$y = 5 - \tan 2(x - \pi)$$

16. RAPPELLING Look back at Example 7. Suppose the cliff is 250 feet high and you are 150 feet from the base. Write and graph an equation that gives your friend's distance from the top as a function of her angle of elevation.

RACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master

skills is on p. 959.

TRANSFORMING GRAPHS Describe how the graph of $y = \sin x$ or $y = \cos x$ can be transformed to produce the graph of the given function.

17.
$$y = 2 + \sin x$$

18.
$$y = 5 - \cos x$$

19.
$$y = -2 + \cos x$$

20.
$$y = \cos\left(x + \frac{\pi}{2}\right)$$

21.
$$y = -\sin(x + \pi)$$

20.
$$y = \cos\left(x + \frac{\pi}{2}\right)$$
 21. $y = -\sin\left(x + \pi\right)$ **22.** $y = \sin\left(x - \frac{\pi}{2}\right)$

23.
$$y = 5 - \cos\left(x - \frac{\pi}{4}\right)$$

24.
$$y = -2 - \sin(x - \pi)$$

23.
$$y = 5 - \cos\left(x - \frac{\pi}{4}\right)$$
 24. $y = -2 - \sin\left(x - \pi\right)$ **25.** $y = 3 + \cos\left(x + \frac{3\pi}{4}\right)$

MATCHING Match the function with its graph.

26.
$$y = -2 + \sin(2x + \pi)$$
 27. $y = -\sin(x + \pi)$

27.
$$y = -\sin(x + \pi)$$

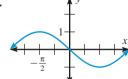
28.
$$y = -3 + \cos x$$

29.
$$y = \cos\left(x + \frac{\pi}{2}\right)$$

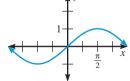
30.
$$y = 1 + \sin \frac{1}{2}x$$

29.
$$y = \cos\left(x + \frac{\pi}{2}\right)$$
 30. $y = 1 + \sin\frac{1}{2}x$ **31.** $y = 1 + 2\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

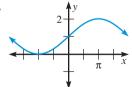
A.



B.



C.

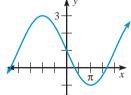


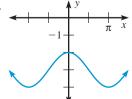
STUDENT HELP

► HOMEWORK HELP

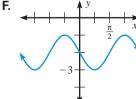
Examples 1-4: Exs. 17-55 **Example 5**: Exs. 57–60 **Example 6:** Exs. 32–55 **Example 7**: Exs. 61, 62











GRAPHING Graph the function.

32.
$$y = 2 + \sin \frac{1}{2}x$$

32.
$$y = 2 + \sin \frac{1}{2}x$$
 33. $y = \cos \left(x - \frac{\pi}{2}\right)$ **34.** $y = -4 \sin \frac{1}{4}x$

34.
$$y = -4 \sin \frac{1}{4} x$$

35.
$$y = 1 + \cos(x + \pi)$$

36.
$$y = -1 + \cos(x - \frac{1}{2})$$

35.
$$y = 1 + \cos(x + \pi)$$
 36. $y = -1 + \cos(x - \pi)$ **37.** $y = 2 + 3\sin(4x + \pi)$

38.
$$y = -4 + \sin 2x$$

38.
$$y = -4 + \sin 2x$$
 39. $y = 1 + 5 \cos (x - \pi)$ **40.** $y = 2 - \sin x$

40.
$$y = 2 - \sin x$$

41.
$$y = \sin\left(x - \frac{3\pi}{2}\right) - 2$$

42.
$$y = \cos\left(2x + \frac{\pi}{2}\right) - 2$$

41.
$$y = \sin\left(x - \frac{3\pi}{2}\right) - 2$$
 42. $y = \cos\left(2x + \frac{\pi}{2}\right) - 2$ **43.** $y = 6 + 4\cos\left(x - \frac{\pi}{2}\right)$

44.
$$y = \frac{1}{2} \tan x$$

STUDENT HELP

HOMEWORK HELP Visit our Web site

www.mcdougallittell.com

for help with Exs. 50-55.

45.
$$y = 2 - \tan\left(x + \frac{\pi}{2}\right)$$
 46. $y = 2 \tan\left(x - \frac{\pi}{2}\right)$

46.
$$y = 2 \tan \left(x - \frac{\pi}{2} \right)$$

47.
$$y = -1 + \tan 2x$$

48.
$$y = 3 + \tan(x - \pi)$$

47.
$$y = -1 + \tan 2x$$
 48. $y = 3 + \tan (x - \pi)$ **49.** $y = 1 + \frac{1}{2} \tan \left(2x - \frac{\pi}{4}\right)$

WRITING EQUATIONS In Exercises 50-55, write an equation of the graph described.

50. The graph of $y = \sin 2\pi x$ translated down 5 units and right 2 units

51. The graph of $y = 3 \cos x$ translated up 3 units and left π units

52. The graph of $y = 5 \tan 4x$ translated left $\frac{\pi}{4}$ unit and then reflected in the x-axis

53. The graph of $y = \frac{1}{3} \sin 6x$ translated down 1 unit and then reflected in the line

54. The graph of $y = \frac{1}{2} \cos \pi x$ translated down $\frac{3}{2}$ units and left 1 unit, and then reflected in the line $y = -\frac{3}{2}$

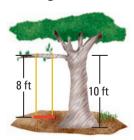
55. The graph of $y = 4 \tan \frac{\pi}{2} x$ translated up 6 units and right $\frac{1}{2}$ unit, and then reflected in the line v = 6

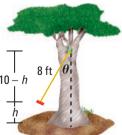
56. CRITICAL THINKING Explain how the graph of $y = \sin x$ can be translated to become the graph of $y = \cos x$.

57. SHEIGHT OF A SWING A swing's height h (in feet) above the ground is

$$h = -8\cos\theta + 10$$

where the pivot is 10 feet above the ground, the rope is 8 feet long, and θ is the angle that the rope makes with the vertical. Graph the function. What is the height of the swing when θ is 45°?





58. Shood Pressure The pressure P (in millimeters of mercury) against the walls of the blood vessels of a certain person is given by

$$P = 100 - 20 \cos \frac{8\pi}{3}t$$

where t is the time (in seconds). Graph the function. If one cycle is equivalent to one heartbeat, what is the person's pulse rate in heartbeats per minute?

S ANIMAL POPULATIONS In Exercises 59

and 60, use the following information. Biologists use sine and cosine functions to model oscillations in predator and prey

populations. The population R of rabbits and the population C of coyotes in a particular region can be modeled by

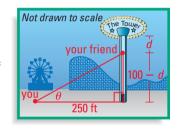
$$R = 25,000 + 15,000 \cos \frac{\pi}{12}t$$
$$C = 5000 + 2000 \sin \frac{\pi}{12}t$$

where *t* is the time in months.

59. Graph both functions in the same coordinate plane and describe how the characteristics of the graphs relate to the diagram shown.



- 61. S AMUSEMENT PARK At an amusement park you watch your friend on a ride that simulates a free fall. You are standing 250 feet from the base of the ride and the ride is 100 feet tall. Write an equation that gives the distance d (in feet) that your friend has fallen as a function of the angle of elevation θ . State the domain of the function. Then graph the function.
- **62. S WINDOW WASHERS** You are standing 80 feet from a 300 foot building, watching as a window washer lowers himself to the ground. Write an equation that gives the window washer's distance d (in feet) from the top of the building as a function of the angle of elevation θ . State the domain of the function. Then graph the function.



Coyote

Rabbit

population

increases.

Coyote population

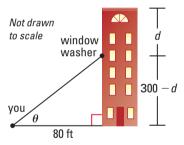
increases.

population

decreases.

Rabbit population

decreases.





- **63. MULTI-STEP PROBLEM** You are at the top of a 120 foot building that straddles a road. You are looking down at a car traveling straight toward the building.
 - **a.** Write an equation of the car's distance from the base of the building as a function of the angle of depression from you to the car.
 - **b.** Suppose the car is between you and a large road sign that you know is one mile (5280 feet) from the building. Write an equation for the distance between the road sign and the car as a function of the angle of depression from you to the car.
 - **c.** Graph the functions you wrote in parts (a) and (b) in the same coordinate plane.
 - **d.** Writing Describe how the graphs you drew in part (c) are geometrically related.



64. SPERRIS WHEEL Suppose a Ferris wheel has a radius of 20 feet and operates at a speed of 3 revolutions per minute. The bottom car is 4 feet above the ground. Write a model for the height of a person above the ground whose height when t = 0 is h = 44.

CLASSIFYING CONICS Classify the conic section and write its equation in standard form. (Review 10.6 for 14.3)

65.
$$36x^2 + 25y^2 - 900 = 0$$

66.
$$9x^2 - 16y^2 - 144 = 0$$

67.
$$10x^2 + 10y^2 - 250 = 0$$

68.
$$100x^2 + 81y^2 - 100 = 0$$

COMBINATIONS Find the number of combinations. (Review 12.2)

69.
$$_{8}C_{7}$$

70.
$$_{142}C_1$$

71.
$${}_{10}C_3$$

72.
$$_{14}C_5$$

73.
$${}_{5}C_{3}$$

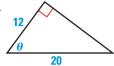
74.
$$_{6}C_{2}$$

75.
$$_{7}C_{6}$$

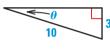
76.
$$_{100}C_2$$

EVALUATING FUNCTIONS Evaluate the six trigonometric functions of the angle θ . (Review 13.1 for 14.3)

77.

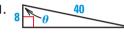


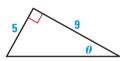




80.







83. Sox Lunches You have made eight different lunches for eight people. How many different ways can you distribute the lunches? (Review 12.1)

Quiz 1

Self-Test for Lessons 14.1 and 14.2

Find the amplitude and period of the function. (Lesson 14.1)

1.
$$y = \frac{5}{2} \sin 7x$$

2.
$$y = \cos 2x$$

$$3. y = \sin \frac{\pi}{2} x$$

4.
$$y = \frac{1}{4} \sin 2\pi x$$
 5. $y = 3 \cos \pi x$ **6.** $y = 4 \cos \frac{3\pi}{2} x$

$$5. y = 3 \cos \pi x$$

6.
$$y = 4 \cos \frac{3\pi}{2} x$$

7.
$$y = \frac{7}{3}\cos 4x$$

8.
$$y = \frac{1}{3} \sin x$$

9.
$$y = 6 \sin \frac{1}{8}x$$

Graph the function. (Lessons 14.1, 14.2)

10.
$$y = 2 \sin \pi x$$

11.
$$y = \frac{3}{2}\cos\frac{1}{2}\pi x$$

12.
$$y = -\sin 2x$$

13.
$$y = 4 \tan \frac{1}{2}x$$

14.
$$y = -4 + 2 \sin 3x$$

14.
$$y = -4 + 2 \sin 3x$$
 15. $y = 3 \tan 2(x + \pi)$

16.
$$y = -3\cos(x + \pi)$$

17.
$$y = -2 + \cos \frac{1}{2}(x - \pi)$$

16.
$$y = -3\cos(x + \pi)$$
 17. $y = -2 + \cos\frac{1}{2}(x - \pi)$ **18.** $y = 2 - 5\tan\left(x + \frac{\pi}{3}\right)$

19. S GLASS ELEVATOR You are standing 120 feet from the base of a 260 foot building. You are looking at your friend who is going up the side of the building in a glass elevator. Write and graph a function that gives your friend's distance d (in feet) above the ground as a function of her angle of elevation θ . What is her angle of elevation when she is 70 feet above the ground? (Lesson 14.2)