13.6 The Law of Cosines

What *you should learn*

GOAL 1 Use the law of cosines to find the sides and angles of a triangle.

GOAL 2 Use Heron's formula to find the area of a triangle, as applied in **Example 5**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding the angle at which two swinging trapeze artists meet in **Ex. 50**.

REAL LIFE

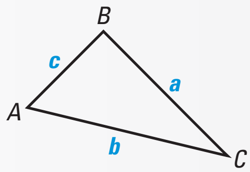
GOAL 1 USING THE LAW OF COSINES

You have not yet solved triangles for which two sides and the included angle (SAS) or three sides (SSS) are given. You can solve both of these cases using the **law of cosines**.

LAW OF COSINES

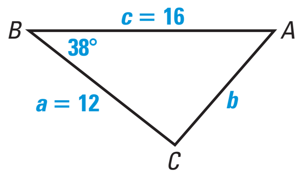
If $\triangle ABC$ has sides of length a, b, and c as shown, then:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$



EXAMPLE 1 *The SAS Case*

Solve $\triangle ABC$ with $a = 12$, $c = 16$, and $B = 38°$.



SOLUTION

Begin by using the law of cosines to find the length b of the third side.

$b^2 = a^2 + c^2 - 2ac \cos B$	Write law of cosines.
$b^2 = 12^2 + 16^2 - 2(12)(16) \cos 38°$	Substitute for a, c, and B.
$b^2 \approx 97.4$	Simplify.
$b \approx \sqrt{97.4} \approx 9.87$	Take square root.

Now that you know all three sides and one angle, you can use the law of cosines *or* the law of sines to find a second angle.

$\frac{\sin A}{a} = \frac{\sin B}{b}$	Write law of sines.
$\frac{\sin A}{12} = \frac{\sin 38°}{9.87}$	Substitute for a, b, and B.
$\sin A = \frac{12 \sin 38°}{9.87}$	Multiply each side by 12.
$\sin A \approx 0.7485$	Simplify.
$A \approx \sin^{-1} 0.7485 \approx 48.5°$	Use inverse sine.

You can find the third angle as follows.

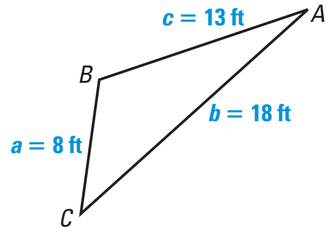
$$C \approx 180° - 38° - 48.5° = 93.5°$$

STUDENT HELP

Derivations
For a derivation of the law of cosines, see p. 901.

EXAMPLE 2 *The SSS Case*

Solve $\triangle ABC$ with $a = 8$ feet, $b = 18$ feet, and $c = 13$ feet.



SOLUTION

First find the angle opposite the longest side, $\overline{AC}$. Using the law of cosines, you can write:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{8^2 + 13^2 - 18^2}{2(8)(13)} = -0.4375$$

Using the inverse cosine function, you can find the measure of obtuse angle B:

$$B = \cos^{-1}(-0.4375) \approx 115.9°$$

Now use the law of sines to find A.

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ **Write law of sines.**

$\dfrac{\sin A}{8} = \dfrac{\sin 115.9°}{18}$ **Substitute.**

$\sin A = \dfrac{8 \sin 115.9°}{18}$ **Multiply each side by 8.**

$\sin A \approx 0.3998$ **Simplify.**

$A \approx \sin^{-1} 0.3998 \approx 23.6°$ **Use inverse sine.**

Finally, you can find the measure of angle C:

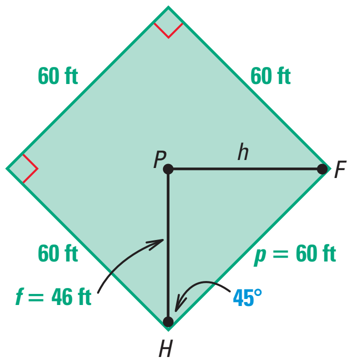
$$C \approx 180° - 23.6° - 115.9° = 40.5°$$

> **STUDENT HELP**
>
> **Study Tip**
> In Example 2 the largest angle is found first to make sure that the other two angles are acute. This way, when you use the law of sines to find another angle measure, you will know that it is between 0° and 90°.

REAL LIFE
Softball

EXAMPLE 3 *The SAS Case*

The pitcher's mound on a softball field is 46 feet from home plate. The distance between the bases is 60 feet. How far is the pitcher's mound from first base?



SOLUTION

Begin by forming $\triangle HPF$. In this triangle you know that $H = 45°$ because the line HP bisects the right angle at home plate. From the given information you know that $f = 46$ and $p = 60$. Using the law of cosines, you can solve for h.

$h^2 = f^2 + p^2 - 2fp \cos H$ **Write law of cosines.**

$h^2 = 46^2 + 60^2 - 2(46)(60) \cos 45°$ **Substitute for *f*, *p*, and *H*.**

$h^2 \approx 1812.8$ **Simplify.**

$h \approx \sqrt{1812.8}$ **Take square root.**

≈ 42.6 feet **Simplify.**

▶ The distance between the pitcher's mound and first base is about 42.6 feet.

GOAL 2 USING HERON'S FORMULA

The law of cosines can be used to establish the following formula for the area of a triangle. This formula is credited to the Greek mathematician Heron (circa A.D. 100).

HERON'S AREA FORMULA

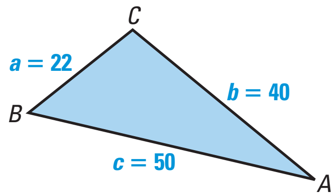
The area of the triangle with sides of length a, b, and c is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$. The variable s is called the *semiperimeter*, or half-perimeter, of the triangle.

EXAMPLE 4 *Finding the Area of a Triangle*

Find the area of $\triangle ABC$.



STUDENT HELP

Look Back
For help with simplifying radical expressions, see p. 264.

SOLUTION

Begin by finding the semiperimeter.

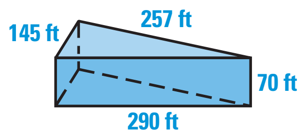
$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(22 + 40 + 50) = 56$$

Now use Heron's formula to find the area of $\triangle ABC$:

$$\begin{aligned}\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{56(56-22)(56-40)(56-50)} \\ &= \sqrt{182{,}784} \approx 428 \text{ square units}\end{aligned}$$

EXAMPLE 5 *Finding the Volume of a Building*

LANDAU BUILDING The dimensions of the Landau Building are given at the right. Find the volume of the building.



FOCUS ON APPLICATIONS

REAL LIFE **LANDAU BUILDING** The Landau Building, located in Cambridge, Massachusetts, was designed by architect I.M. Pei. Pei's work is known to have a sharp, geometric look.

SOLUTION

Begin by finding the area of the base. The semiperimeter of the base is:

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(145 + 257 + 290) = 346$$

So, the area of the base is:

$$\begin{aligned}\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{346(346-145)(346-257)(346-290)} \\ &\approx 18{,}600 \text{ square feet}\end{aligned}$$

To find the volume, multiply this area by the building's height:

$$\text{Volume} = (\text{Area of base})(\text{Height}) \approx (18{,}600)(70) = 1{,}302{,}000 \text{ cubic feet}$$

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: In a triangle with sides of length a, b, and c, $\frac{1}{2}(a + b + c)$ is called the __?__.

Concept Check ✓

2. For each case, tell whether you would use the *law of sines* or the *law of cosines* to solve the triangle.

a. SSS **b.** SSA **c.** SAS **d.** ASA **e.** AAS

3. If when using the law of cosines to find angle A in $\triangle ABC$, you get $\cos A < 0$, what type of angle is A?

4. Express Heron's formula in words.

Skill Check ✓

Solve $\triangle ABC$.

5. $B = 20°, a = 120, c = 100$

6. $C = 95°, a = 10, b = 12$

7. $a = 25, b = 11, c = 24$

8. $a = 2, b = 4, c = 5$

Find the area of $\triangle ABC$ having the given side lengths.

9. $a = 25, b = 60, c = 45$

10. $a = 9, b = 4, c = 11$

11. $a = 100, b = 55, c = 61$

12. $a = 5, b = 27, c = 29$

BASEBALL In Exercises 13 and 14, use the following information. The pitcher's mound on a baseball field is 60.5 feet from home plate. The distance between the bases is 90 feet.

13. How far is the pitcher's mound from first base?

14. Using Heron's formula, find the area of the triangle formed by the pitcher's mound, home plate, and first base.

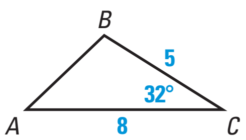
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 959.

SOLVING TRIANGLES **Solve $\triangle ABC$.**

15.



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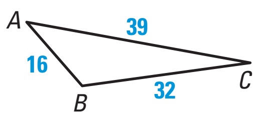

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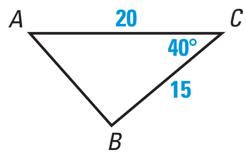


SOLVING TRIANGLES **Solve $\triangle ABC$.**

18. $B = 20°, a = 120, c = 100$

19. $C = 95°, a = 10, b = 12$

20. $a = 25, b = 11, c = 24$

21. $a = 2, b = 4, c = 5$

22. $A = 78°, b = 2, c = 4$

23. $A = 60°, b = 30, c = 28$

24. $B = 45°, a = 11, c = 22$

25. $C = 30°, a = 20, b = 20$

26. $a = 9, b = 3, c = 11$

27. $B = 15°, a = 12, c = 6$

28. $a = 25, b = 26, c = 5$

29. $a = 47, b = 30, c = 62$

STUDENT HELP

HOMEWORK HELP
Examples 1, 2: Exs. 15–37
Example 3: Exs. 50, 51
Example 4: Exs. 38–48
Example 5: Exs. 52–54

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 30–37.

CHOOSING A METHOD **Use the law of sines, the law of cosines, or the Pythagorean theorem to solve $\triangle ABC$.**

30. $A = 96°, B = 39°, b = 13$

31. $B = 80°, C = 30°, b = 34$

32. $A = 34°, b = 17, c = 48$

33. $C = 104°, b = 11, c = 32$

34. $A = 48°, B = 51°, c = 36$

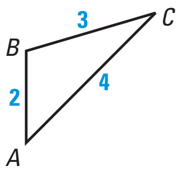
35. $a = 48, b = 51, c = 36$

36. $B = 10°, b = 5, c = 25$

37. $C = 90°, a = 4, b = 11$

FINDING AREA **Find the area of $\triangle ABC$.**

38.



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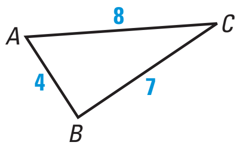

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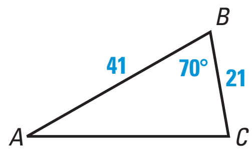


FINDING AREA **Find the area of $\triangle ABC$ having the given side lengths.**

41. $a = 15, b = 20, c = 25$

42. $a = 13, b = 10, c = 4$

43. $a = 75, b = 68, c = 72$

44. $a = 3, b = 19, c = 21$

45. $a = 4, b = 2, c = 4$

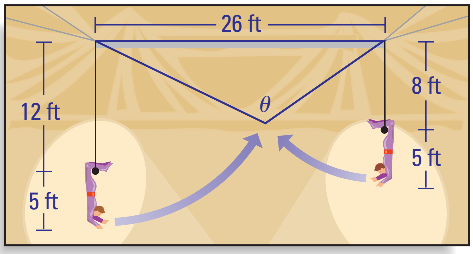
46. $a = 20, b = 21, c = 37$

47. $a = 8, b = 8, c = 8$

48. $a = 18, b = 15, c = 10$

49. **CRITICAL THINKING** Explain why the Pythagorean theorem is a special case of the law of cosines.

50. **TRAPEZE ARTISTS** The diagram shows the path of two trapeze artists who are both 5 feet long when hanging by their knees. The "flyer" on the left bar is preparing to make hand-to-hand contact with the "catcher" on the right bar. At what angle θ will the two meet? ▶ Source: Trapeze Arts, Inc.



FOCUS ON CAREERS

REAL LIFE SURVEYOR
A surveyor takes precise measurements to establish official land airspace, and water boundaries. A surveyor often uses an instrument called a *transit*, as pictured above, to measure angles.

INTERNET CAREER LINK
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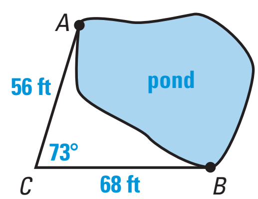
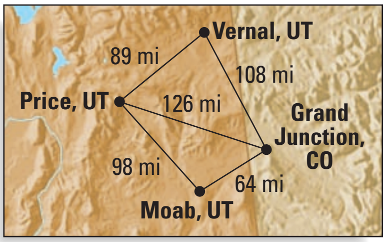
51. **SURVEYING** You are a surveyor measuring the width of a pond from point A to point B, as shown. You set up your transit at point C and measure an angle of 73°. You also measure the distance from point C to points A and B, getting 56 feet and 68 feet, respectively. What is the width of the pond?



52. **GIBSON BLOCK** Built in 1913, the Gibson Block in Alberta, Canada, is shaped like a flat clothing iron of that time period. The approximately triangular base of the building has sides of length 18.3 meters, 37.1 meters, and 41.0 meters. The height of the Gibson Block is about 13.2 meters. Find the volume of the Gibson Block. ▶ Source: Stantec Architecture Ltd.

53. **DINOSAUR DIAMOND** In Utah and Colorado, an area called the Dinosaur Diamond is known for containing many dinosaur fossils. The map at the right shows the towns at the four vertices of the diamond. Use the given distances to find the area of the Dinosaur Diamond.

▶ Source: Dinomation



54. **FERTILIZER** A farmer has a triangular field with sides that are 240 feet, 300 feet, and 360 feet long. He wants to apply fall fertilizer to the field. If it takes one 40 pound bag of fertilizer to cover 6000 square feet, how many bags does he need to cover the field?

Test Preparation

55. **MULTIPLE CHOICE** Two airplanes leave an airport at the same time, the first headed due north and the second headed 37° east of north. At 2:00 P.M. the first airplane is 250 miles from the airport and the second airplane is 316 miles from the airport. How far apart are the two airplanes?

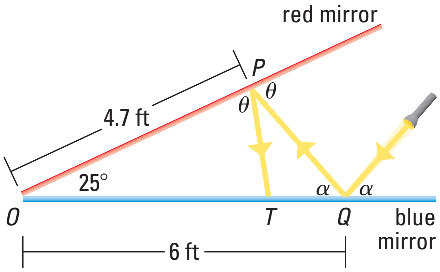
Ⓐ about 190 miles Ⓑ about 210 miles Ⓒ about 200 miles
Ⓓ about 310 miles Ⓔ about 165 miles

56. **MULTIPLE CHOICE** Find the area of a triangle with sides of length 37 feet, 23 feet, and 42 feet.

Ⓐ about 189 ft^2 Ⓑ about 134 ft^2 Ⓒ about 477 ft^2
Ⓓ about 424 ft^2 Ⓔ about 777 ft^2

★ Challenge

57. **MIRRORS** In the diagram, a beam of light is directed at the blue mirror, reflected to the red mirror, and then reflected back to the blue mirror. Find the distance PT that the light travels from the red mirror back to the blue mirror given that $OQ = 6$ feet and $OP = 4.7$ feet. (*Hint:* You will need to find θ. To do this, find $\angle OPQ$ and use the fact that $2\theta + m\angle TPQ = 180°$.)



EXTRA CHALLENGE
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MIXED REVIEW

WRITING EQUATIONS Write an equation of the hyperbola with the given foci and vertices. (Review 10.5)

58. Foci: $(-7, 0), (7, 0)$
Vertices: $(-2, 0), (2, 0)$

59. Foci: $(0, -11), (0, 11)$
Vertices: $(0, -3), (0, 3)$

60. Foci: $(-9, 0), (9, 0)$
Vertices: $(-5, 0), (5, 0)$

61. Foci: $\left(0, -2\sqrt{5}\right), \left(0, 2\sqrt{5}\right)$
Vertices: $(0, -1), (0, 1)$

CALCULATING PROBABILITIES Calculate the probability of rolling a die 30 times and getting the given number of 4's. (Review 12.6)

62. 1 **63.** 3 **64.** 5 **65.** 6 **66.** 8 **67.** 10

68. **VERTICAL MOTION** From a height of 120 feet, how long does it take a ball thrown downward at 20 feet per second to hit the ground? (Review 5.6 for 13.7)