13.5

The Law of Sines

GOAL 1 USING THE LAW OF SINES

In Lesson 13.1 you learned how to solve right triangles. To solve a triangle with no right angle, you need to know the measure of at least one side and any two other parts of the triangle. This breaks down into four possible cases.

1. Two angles and any side (AAS or ASA)

2. Two sides and an angle opposite one of them (SSA)

3. Three sides (SSS)
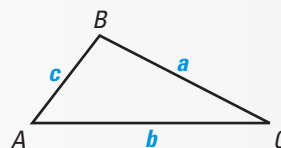
4. Two sides and their included angle (SAS)

The first two cases can be solved using the **law of sines**. The last two cases require the law of cosines, which you will study in Lesson 13.6.

LAW OF SINES

If △ABC has sides of length a, b, and c as shown, then:
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

An equivalent form is $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

EXAMPLE 1   *The AAS or ASA Case*

Solve △ABC with $C = 103°$, $B = 28°$, and $b = 26$ feet.

SOLUTION

You can find the third angle of △ABC as follows.

$$A = 180° - 103° - 28° = 49°$$

By the law of sines you can write:

$$\frac{a}{\sin 49°} = \frac{26}{\sin 28°} = \frac{c}{\sin 103°}$$

You can then solve for a and c as follows.

$\frac{a}{\sin 49°} = \frac{26}{\sin 28°}$	**Write two equations, each with one variable.**	$\frac{c}{\sin 103°} = \frac{26}{\sin 28°}$
$a = \frac{26 \sin 49°}{\sin 28°}$	**Solve for the variable.**	$c = \frac{26 \sin 103°}{\sin 28°}$
$a \approx 41.8$ feet	**Use a calculator.**	$c \approx 54.0$ feet
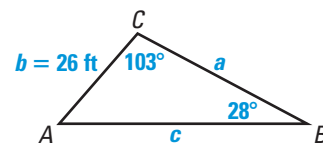
Two angles and one side (AAS or ASA) determine exactly one triangle. Two sides and an angle opposite one of those sides (SSA) may determine no triangle, one triangle, or two triangles. The SSA case is called the *ambiguous case*.
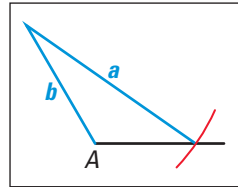

POSSIBLE TRIANGLES IN THE SSA CASE

Consider a triangle in which you are given *a, b,* and *A.*

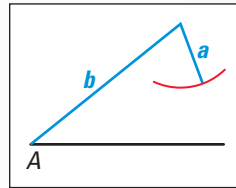| **A IS OBTUSE.**	**A IS ACUTE.**

A IS OBTUSE.
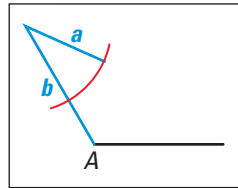
$a \le b$
No triangle

$a > b$
One triangle

A IS ACUTE.

$b \sin A > a$
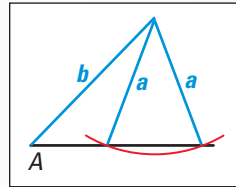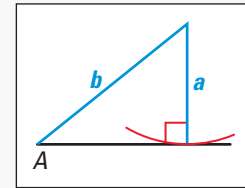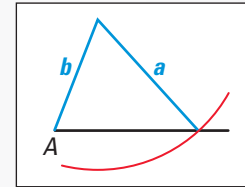No triangle

$b \sin A = a$
One triangle

$b \sin A < a < b$
Two triangles

$a > b$
One triangle

EXAMPLE 2 *The SSA Case—One Triangle*

Solve $\triangle ABC$ with $C = 122°$, $a = 12$ cm, and $c = 18$ cm.

SOLUTION

First make a sketch. Because C is obtuse and the side opposite C is longer than the given adjacent side, you know that only one triangle can be formed. Use the law of sines to find A.

$$\frac{\sin A}{12} = \frac{\sin 122°}{18} \qquad \text{Law of sines}$$

$$\sin A = \frac{12 \sin 122°}{18} \qquad \text{Multiply each side by 12.}$$

$$\sin A \approx 0.5654 \qquad \text{Use a calculator.}$$

$$A \approx 34.4° \qquad \text{Use inverse sine function.}$$

You then know that $B \approx 180° - 122° - 34.4° = 23.6°$. Use the law of sines again to find the remaining side length b of the triangle.

$$\frac{b}{\sin 23.6°} = \frac{18}{\sin 122°}$$

$$b = \frac{18 \sin 23.6°}{\sin 122°} \approx 8.5 \text{ centimeters}$$
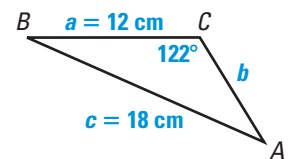
EXAMPLE 3 — *The SSA Case—No Triangle*

Solve $\triangle ABC$ with $a = 4$ inches, $b = 2.5$ inches, and $B = 58°$.

SOLUTION

Begin by drawing a horizontal line. On one end form a 58° angle (B) and draw a segment ($\overline{BC}$) 4 inches long. At vertex C, use a compass to draw an arc of radius 2.5 inches. This arc does not intersect the horizontal line, so it is not possible to draw the indicated triangle.

You can see that b needs to be at least $4 \sin 58° \approx 3.39$ inches long to reach the horizontal side and form a triangle.
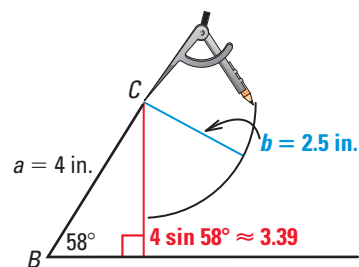
EXAMPLE 4 — *The SSA Case—Two Triangles*

ASTRONOMY At certain times during the year, you can see Venus in the morning sky. The distance between Venus and the sun is approximately 67 million miles. The distance between Earth and the sun is approximately 93 million miles. Estimate the distance between Venus and Earth if the observed angle between the sun and Venus is 34°.

SOLUTION
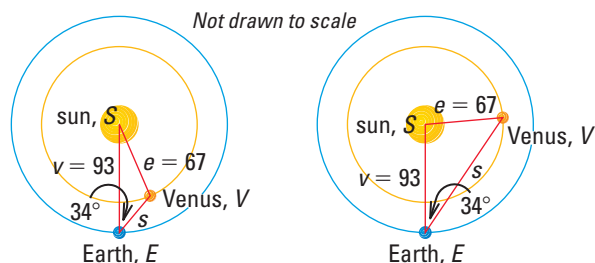
Venus's distance from the sun, $e = 67$, is greater than $v \sin E = 93 \sin 34° \approx 52$ and less than Earth's distance from the sun, $v = 93$. Therefore, two possible triangles can be formed. Draw diagrams as shown. Use the law of sines to find the possible measures of V.

$$\frac{\sin 34°}{67} = \frac{\sin V}{93}$$

$$\sin V = \frac{93 \sin 34°}{67}$$

$$\sin V \approx 0.7762$$

Not drawn to scale

There are two angles between 0° and 180° for which $\sin V \approx 0.7762$. Use your calculator to find the angle between 0° and 90°: $\sin^{-1} 0.7762 \approx 50.9°$. To find the second angle, subtract the angle given by your calculator from 180°: $180° - 50.9° = 129.1°$. So, $V \approx 50.9°$ or $V \approx 129.1°$.

Because the sum of the angle measures in a triangle equals 180°, you know that $S \approx 95.1°$ when $V \approx 50.9°$ or $S \approx 16.9°$ when $V \approx 129.1°$. Finally, use the law of sines again to find the side length s.

$$\frac{s}{\sin 95.1°} = \frac{67}{\sin 34°} \qquad\qquad \frac{s}{\sin 16.9°} = \frac{67}{\sin 34°}$$

$$s = \frac{67 \sin 95.1°}{\sin 34°} \qquad\qquad s = \frac{67 \sin 16.9°}{\sin 34°}$$

$$\approx 119 \qquad\qquad\qquad\quad \approx 34.8$$

▶ The approximate distance between Venus and Earth is either 119 million miles or 34.8 million miles.
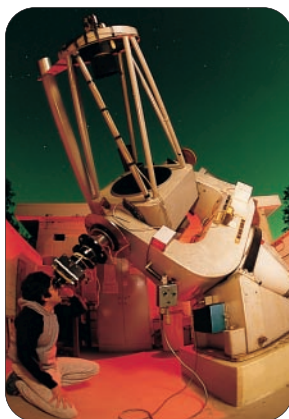
GOAL 2 FINDING THE AREA OF A TRIANGLE

You can find the area of any triangle if you know the lengths of two sides and the measure of the included angle.

AREA OF A TRIANGLE

The area of any triangle is given by one half the product of the lengths of two sides times the sine of their included angle. For $\triangle ABC$ shown, there are three ways to calculate the area:
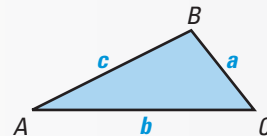
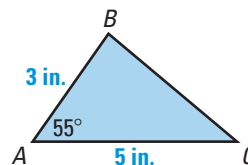$$\text{Area} = \frac{1}{2}bc \sin A \qquad \text{Area} = \frac{1}{2}ac \sin B \qquad \text{Area} = \frac{1}{2}ab \sin C$$

EXAMPLE 5 *Finding a Triangle's Area*

Find the area of $\triangle ABC$.

SOLUTION
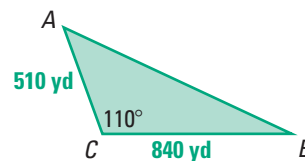
Use the appropriate formula for the area of a triangle.

$$\text{Area} = \frac{1}{2}bc \sin A$$

$$= \frac{1}{2}(5)(3) \sin 55°$$

$$\approx 6.14 \text{ square inches}$$

Real Estate

EXAMPLE 6 *Calculating the Price of Land*

You are buying the triangular piece of land shown. The price of the land is $2000 per acre (1 acre = 4840 square yards). How much does the land cost?

SOLUTION

The area of the land is:

$$\text{Area} = \frac{1}{2}ab \sin C$$

$$= \frac{1}{2}(840)(510) \sin 110°$$

$$\approx 201{,}000 \text{ square yards}$$

▶ The property contains $201{,}000 \div 4840 \approx 41.5$ acres. At $2000 per acre, the price of the land is about $(2000)(41.5) = \$83{,}000$.

GUIDED PRACTICE

Vocabulary Check ✓ 1. What is the SSA case called? Why is it called this?

Concept Check ✓ 2. Which two of the following cases can be solved using the law of sines?

     A. SSS   **B.** SSA   **C.** AAS or ASA   **D.** SAS

  3. Suppose a, b, and A are given for $\triangle ABC$ where $A < 90°$. Under what conditions would you have no triangle? one triangle? two triangles?

Skill Check ✓ **Decide whether the given measurements can form exactly *one triangle*, exactly *two triangles*, or *no triangle*. (You do not need to solve the triangle.)**

  4. $C = 65°$, $c = 44$, $b = 32$ 5. $A = 140°$, $a = 5$, $c = 7$

  6. $A = 18°$, $a = 16$, $c = 10$ 7. $A = 70°$, $a = 155$, $c = 160$

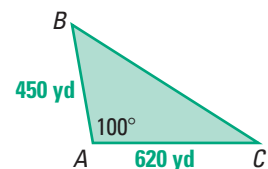**Solve $\triangle ABC$.**

8.

9.

10.
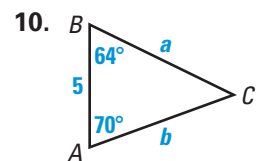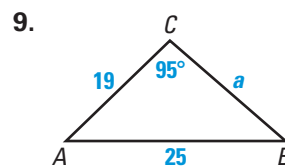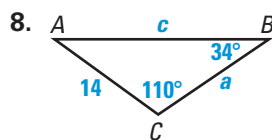
Find the area of the triangle with the given side lengths and included angle.

  11. $b = 2$, $c = 3$, $A = 47°$ 12. $a = 23$, $b = 15$, $C = 51°$

  13. $a = 13$, $c = 24$, $B = 127°$ 14. $b = 12$, $c = 17$, $A = 103°$

  15. 🌐 **REAL ESTATE** Suppose you are buying the triangular piece of land shown. The price of the land is $2200 per acre (1 acre = 4840 square yards). How much does the land cost?

PRACTICE AND APPLICATIONS

NUMBER OF SOLUTIONS Decide whether the given measurements can form exactly *one triangle*, exactly *two triangles*, or *no triangle*.

  16. $C = 65°$, $c = 44$, $b = 32$ 17. $A = 140°$, $a = 5$, $c = 7$

  18. $A = 18°$, $a = 16$, $c = 10$ 19. $A = 70°$, $a = 155$, $c = 160$

  20. $C = 160°$, $c = 12$, $b = 15$ 21. $B = 105°$, $b = 11$, $a = 5$

  22. $B = 56°$, $b = 13$, $a = 14$ 23. $C = 25°$, $c = 6$, $b = 20$

SOLVING TRIANGLES Solve $\triangle ABC$.
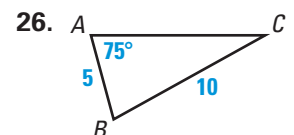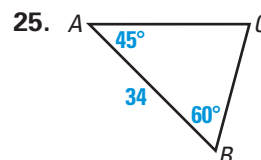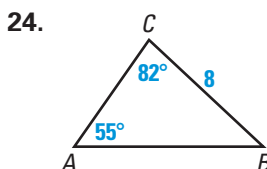
24.

25.

26.
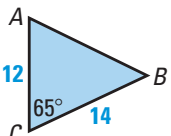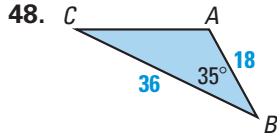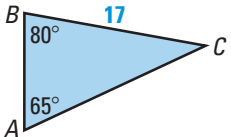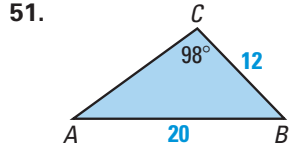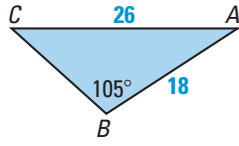
SOLVING TRIANGLES Solve △ABC. (*Hint:* Some of the "triangles" have no solution and some have two solutions.)

27. $B = 60°$, $b = 30$, $c = 20$ **28.** $B = 110°$, $C = 30°$, $a = 15$

29. $B = 130°$, $a = 10$, $b = 8$ **30.** $A = 20°$, $a = 10$, $c = 11$

31. $C = 95°$, $a = 8$, $c = 9$ **32.** $A = 70°$, $B = 60°$, $c = 25$

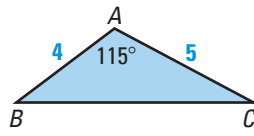**33.** $C = 16°$, $b = 92$, $c = 32$ **34.** $A = 10°$, $C = 130°$, $b = 5$

35. $B = 35°$, $a = 12$, $b = 26$ **36.** $C = 145°$, $b = 5$, $c = 9$

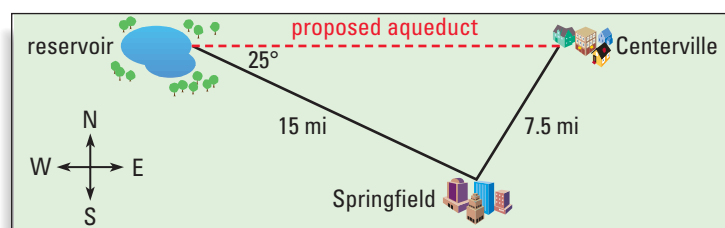**FINDING AREA** Find the area of the triangle with the given side lengths and included angle.

37. $B = 25°$, $a = 17$, $c = 33$ **38.** $C = 130°$, $a = 21$, $b = 17$

39. $C = 120°$, $a = 8$, $b = 5$ **40.** $A = 85°$, $b = 11$, $c = 18$

41. $A = 75°$, $b = 16$, $c = 21$ **42.** $B = 110°$, $a = 11$, $c = 24$

43. $C = 125°$, $a = 3$, $b = 8$ **44.** $B = 29°$, $a = 13$, $c = 13$

45. $B = 96°$, $a = 15$, $c = 9$ **46.** $A = 32°$, $b = 10$, $c = 12$

FINDING AREA Find the area of △ABC.

47.

**48.**

49.

50.

51.

52.

FINDING A PATTERN In Exercises 53–55, use a graphing calculator to explore how the angle measure between two sides of a triangle affects the area of the triangle.

53. Choose a fixed length for each of two sides of a triangle. Let x represent the measure of the included angle. Enter an equation for the area of this triangle into the calculator.

54. Use the *Table* feature to look at the y-values for $0° < x < 180°$. Does area always increase for increasing values of x? Explain.

55. What value of x maximizes area?

56. 🌎 **AQUEDUCT** A reservoir supplies water through an aqueduct to Springfield, which is 15 miles from the reservoir at 25° south of east. A pumping station at Springfield pumps water 7.5 miles to Centerville, which is due east from the reservoir. Plans have been made to build an aqueduct directly from the reservoir to Centerville. How long will the aqueduct be?

In Exercises 57–59, use the following information.

In 1802 Captain William Lambton began what is known as the Great Trigonometrical Survey of India. Lambton and his company systematically divided India into triangles. They used trigonometry to find unknown distances from a known distance they measured, called a *baseline*. The map below shows a section of the Great Trigonometrical Survey of India.
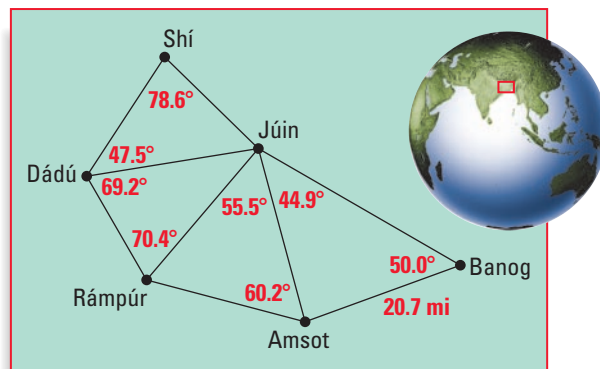
57. Use the given measurements to find the distance between Júin and Amsot.

58. Find the distance between Júin and Rámpúr.

59. *Writing* How could you find the distance from Shí to Dádú? Explain.

60. 🌐 **NEW YORK CITY** You are on the observation deck of the Empire State Building looking at the Chrysler Building. When you turn about 145° you see the Statue of Liberty. You know that the Chrysler Building and the Empire State Building are about 0.6 mile apart and that the Chrysler Building is about 5.7 miles from the Statue of Liberty. Find the approximate distance between the Empire State Building and the Statue of Liberty.

ART CONNECTION **In Exercises 61 and 62, use the following information.**

You are creating a sculpture for an art show at your school. One 50 inch wooden beam makes an angle of 70° with the base of your sculpture. You have another wooden beam 48 inches long that you would like to attach to the top of the 50 inch beam and to the base of the sculpture, as shown below.
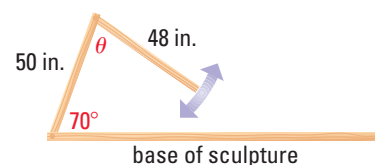
61. Find all possible angles θ that the 48 inch beam can make with the 50 inch beam.

62. Find all possible distances d that the bottom of the 48 inch beam can be from the left end of the base.


63. 🌐 **HANG GLIDER** A hang glider is shown at the right. Use the given nose angle and wing measurements to approximate the area of the sail.
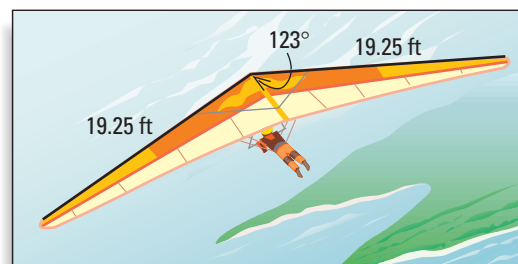
🌐 **COURTYARD** **In Exercises 64 and 65, use the following information.**

You are seeding a triangular courtyard. One side of the courtyard is 52 feet long and another side is 46 feet long. The angle opposite the 52 foot side is 65°.

64. How long is the third side of the courtyard?

65. One bag of grass seed covers an area of 50 square feet. How many bags of grass seed will you need to cover the courtyard?
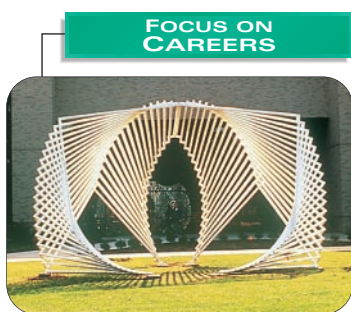
🌎 **BUYING PAINT** In Exercises 66 and 67, use the following information.
You plan to paint the side of the house shown below. One gallon of paint will cover an area of 400 square feet.
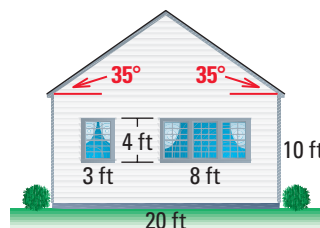
66. Find the area to be painted. Do not include the window area.

67. How many gallons of paint do you need?

Test Preparation

68. **MULTI-STEP PROBLEM** You are at an unknown distance d from a mountain, as shown below. The angle of elevation to the top of the mountain is 65°. You step back 100 feet and measure the angle of elevation to be 60°.

   a. Find the height h of the mountain using the law of sines and right triangle trigonometry. (*Hint*: First find θ.)

   b. Find the height h of the mountain using a system of equations. Set up one tangent equation involving the ratio of d and h, and another tangent equation involving the ratio of $100 + d$ and h, and then solve the system.

   c. ✐ *Writing* Which method was easier for you to use? Explain.
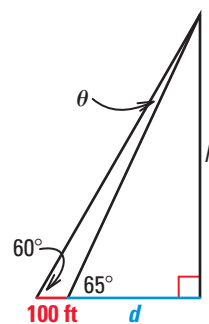
★ **Challenge**

69. **DERIVING FORMULAS** Using the triangle shown at the right as a reference, derive the formulas for the area of a triangle given in the property box on page 802. Then show how to derive the law of sines using the area formulas.
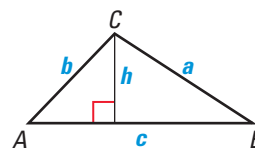
MIXED REVIEW

COMBINING EXPRESSIONS Perform the indicated operations. **(Review 7.2 for 13.6)**

70. $5\sqrt{11} + \sqrt{11} - 9\sqrt{11}$

71. $2\sqrt{12} + 5\sqrt{12} + 3\sqrt{27}$

72. $\sqrt{125} - 7\sqrt{45} + 10\sqrt{40}$

73. $\sqrt{7} + 5\sqrt{63} - 2\sqrt{112}$

74. $2\sqrt{486} - 5\sqrt{54} - 2\sqrt{150}$

75. $\sqrt{72} + 6\sqrt{98} - 10\sqrt{8}$

FINDING COSINE VALUES Use a calculator to evaluate the trigonometric function. Round the result to four decimal places. **(Review 13.1, 13.3 for 13.6)**

76. $\cos 52°$

77. $\cos \dfrac{12\pi}{5}$

78. $\cos \dfrac{9\pi}{5}$

79. $\cos \dfrac{10\pi}{7}$

80. $\cos 20°$

81. $\cos 305°$

82. $\cos (-200°)$

83. $\cos 5°$

84. 🌎 **CAR HEADLIGHTS** In Massachusetts the low-beam headlights of cars are set to focus down 4 inches at a distance of 10 feet. At what angle θ are the beams directed? **(Review 13.4)**