13.4 Inverse Trigonometric Functions

What you should learn

GOAL 1 Evaluate inverse trigonometric functions.

GOAL 2 Use inverse trigonometric functions to solve **real-life** problems, such as finding an angle of repose in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the angle at which to set the arm of a crane in **Example 5**.

REAL LIFE

GOAL 1 EVALUATING AN INVERSE TRIGONOMETRIC FUNCTION

In the first three lessons of this chapter, you learned to evaluate trigonometric functions of a given angle. In this lesson you will study the reverse problem—finding angles that correspond to a given value of a trigonometric function.

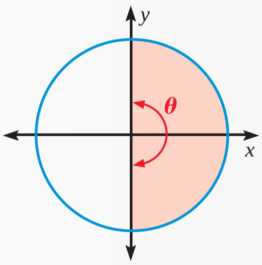
Suppose you were asked to find an angle θ whose sine is 0.5. After thinking about the problem for a while, you would probably realize that there are *many* such angles. For instance, the angles

$$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \text{ and } -\frac{7\pi}{6}$$

all have a sine value of 0.5. (Try checking this with a calculator.) Of these, the value of the *inverse sine function* at 0.5 is defined to be $\frac{\pi}{6}$. General definitions of inverse sine, inverse cosine, and inverse tangent are given below.

INVERSE TRIGONOMETRIC FUNCTIONS

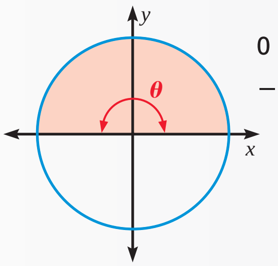
- If $-1 \le a \le 1$, then the **inverse sine** of a is $\sin^{-1} a = \theta$ where $\sin \theta = a$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ (or $-90° \le \theta \le 90°$).



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$-1 \le \sin \theta \le 1$

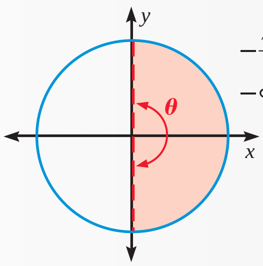
- If $-1 \le a \le 1$, then the **inverse cosine** of a is $\cos^{-1} a = \theta$ where $\cos \theta = a$ and $0 \le \theta \le \pi$ (or $0° \le \theta \le 180°$).



$0 \le \theta \le \pi$

$-1 \le \cos \theta \le 1$

- If a is any real number, then the **inverse tangent** of a is $\tan^{-1} a = \theta$ where $\tan \theta = a$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (or $-90° < \theta < 90°$).



$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$-\infty < \tan \theta < +\infty$

EXAMPLE 1 *Evaluating Inverse Trigonometric Functions*

Evaluate the expression in both radians and degrees.

a. $\sin^{-1} \frac{\sqrt{3}}{2}$ **b.** $\cos^{-1} 2$ **c.** $\tan^{-1} (-1)$

SOLUTION

a. When $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, or $-90° \le \theta \le 90°$, the angle whose sine is $\frac{\sqrt{3}}{2}$ is:

$$\theta = \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3} \qquad \text{or} \qquad \theta = \sin^{-1} \frac{\sqrt{3}}{2} = 60°$$

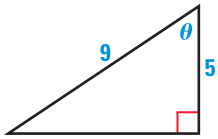
b. There is no angle whose cosine is 2. So, $\cos^{-1} 2$ is undefined.

c. When $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, or $-90° < \theta < 90°$, the angle whose tangent is -1 is:

$$\theta = \tan^{-1} (-1) = -\frac{\pi}{4} \qquad \text{or} \qquad \theta = \tan^{-1} (-1) = -45°$$

EXAMPLE 2 *Finding an Angle Measure*

Find the measure of the angle θ for the triangle shown.



SOLUTION

In the right triangle, you are given the adjacent side and the hypotenuse. You can write:

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{9}$$

This equation is asking you to find the acute angle whose cosine is $\frac{5}{9}$. Use a calculator to find the measure of θ.

$$\theta = \cos^{-1} \frac{5}{9} \approx 0.982 \text{ radians} \qquad \text{or} \qquad \theta = \cos^{-1} \frac{5}{9} \approx 56.3°$$

STUDENT HELP

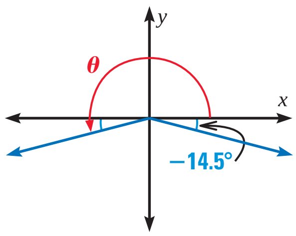
Study Tip
When approximating the value of an angle, make sure your calculator is set to radian mode if you want your answer in radians, or to degree mode if you want your answer in degrees.

EXAMPLE 3 *Solving a Trigonometric Equation*

Solve the equation $\sin \theta = -\frac{1}{4}$ where $180° < \theta < 270°$.

SOLUTION

In the interval $-90° < \theta < 90°$, the angle whose sine is $-\frac{1}{4}$ is $\sin^{-1}\left(-\frac{1}{4}\right) \approx -14.5°$. This angle is in Quadrant IV as shown. In Quadrant III (where $180° < \theta < 270°$), the angle that has the same sine value is:



$$\theta \approx 180° + 14.5° = 194.5°$$

✓**CHECK** Use a calculator to check the answer.

$$\sin 194.5° \approx -0.25 \checkmark$$

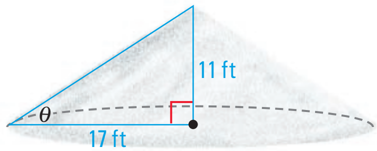
FOCUS ON APPLICATIONS

ROCK SALT Each year about 9 million tons of rock salt are poured on highways in North America to melt ice. Although rock salt is the best deicing material, it also eats away at cars and road surfaces.

GOAL 2 Using Inverse Trigonometric Functions in Real Life

EXAMPLE 4 *Writing and Solving a Trigonometric Equation*

ROCK SALT Different types of granular substances naturally settle at different angles when stored in cone-shaped piles. This angle θ is called the *angle of repose*. When rock salt is stored in a cone-shaped pile 11 feet high, the diameter of the pile's base is about 34 feet. ▶ Source: Bulk-Store Structures, Inc.



a. Find the angle of repose for rock salt.

b. How tall is a pile of rock salt that has a base diameter of 50 feet?

SOLUTION

a. In the right triangle shown inside the cone, you are given the opposite side and the adjacent side. You can write:

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{11}{17}$$

This equation is asking you to find the acute angle whose tangent is $\frac{11}{17}$.

$$\theta = \tan^{-1}\frac{11}{17} \approx 32.9°$$

▶ The angle of repose for rock salt is about 32.9°.

b. The pile of rock salt has a base radius of 25 feet. From part (a) you know that the angle of repose for rock salt is about 32.9°. To find the height h (in feet) of the pile you can write:

$$\tan 32.9° = \frac{h}{25}$$

$$h = 25\tan 32.9°$$

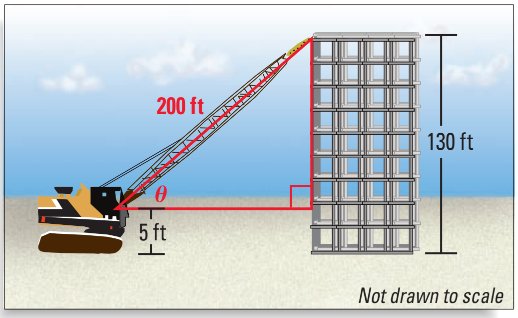
$$\approx 16.2$$

▶ The pile of rock salt is about 16.2 feet tall.

REAL LIFE *Construction*

EXAMPLE 5 *Writing and Solving a Trigonometric Equation*

A crane has a 200 foot arm whose lower end is 5 feet off the ground. The arm has to reach the top of a building 130 feet high. At what angle θ should the arm be set?



SOLUTION

In the right triangle in the diagram, you know the opposite side and the hypotenuse. You can write:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{130 - 5}{200} = \frac{5}{8}$$

This equation is asking you to find the acute angle whose sine is $\frac{5}{8}$.

$$\theta = \sin^{-1}\frac{5}{8} \approx 38.7°$$

▶ The crane's arm should be set at an acute angle of about 38.7°.

Guided Practice

Vocabulary Check ✓

1. Complete this statement: The $\underline{\ ?\ }$ sine of 1 equals $\frac{\pi}{2}$, or 90°.

Concept Check ✓

2. Explain why the domain of $y = \cos\theta$ cannot be restricted to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ if the inverse is to be a function.

3. Explain why $\tan^{-1} 3$ is defined, but $\cos^{-1} 3$ is undefined.

4. **Error Analysis** A student needed to find an angle θ in Quadrant III such that $\sin\theta = -0.3221$. She used a calculator to find that $\sin^{-1}(-0.3221) \approx -18.8°$. Then she added this result to 180° to get an answer of $\theta = 161.2°$. What did she do wrong?

Skill Check ✓

Evaluate the expression without using a calculator.

5. $\tan^{-1}\sqrt{3}$
6. $\cos^{-1}\frac{\sqrt{2}}{2}$
7. $\sin^{-1}\frac{1}{2}$
8. $\cos^{-1}\left(-\frac{1}{2}\right)$

Use a calculator to evaluate the expression in both radians and degrees. Round to three significant digits.

9. $\tan^{-1} 3.9$
10. $\cos^{-1}(-0.94)$
11. $\cos^{-1} 0.34$
12. $\sin^{-1}(-0.4)$

Solve the equation for θ. Round to three significant digits.

13. $\sin\theta = -0.35$; $180° < \theta < 270°$
14. $\tan\theta = 2.4$; $180° < \theta < 270°$
15. $\cos\theta = 0.43$; $270° < \theta < 360°$
16. $\sin\theta = 0.8$; $90° < \theta < 180°$

17. **Construction** A crane has a 150 foot arm whose lower end is 4 feet off the ground. The arm has to reach the top of a building 105 feet high. At what angle should the crane's arm be set?

Practice and Applications

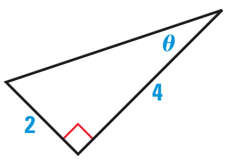
Student Help

Extra Practice to help you master skills is on p. 958.

Evaluating Expressions **Evaluate the expression without using a calculator. Give your answer in both radians and degrees.**

18. $\sin^{-1}\frac{\sqrt{2}}{2}$
19. $\cos^{-1}\frac{1}{2}$
20. $\tan^{-1} 1$
21. $\sin^{-1} 0$
22. $\cos^{-1}(-1)$
23. $\sin^{-1}(-1)$
24. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
25. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Finding Angles **Find the measure of the angle θ. Round to three significant digits.**

26.


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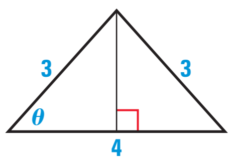

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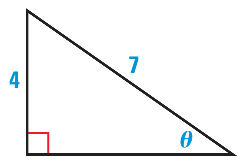

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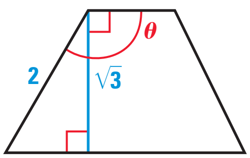

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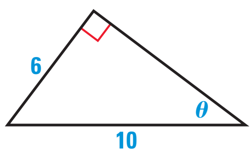
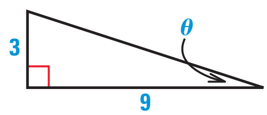


31.


Student Help

Homework Help
Example 1: Exs. 18–25, 32–43
Example 2: Exs. 26–43
Example 3: Exs. 44–51
Examples 4, 5: Exs. 52–57

EVALUATING EXPRESSIONS **Use a calculator to evaluate the expression in both radians and degrees. Round to three significant digits.**

32. $\tan^{-1} 3.9$ **33.** $\cos^{-1} 0.24$ **34.** $\cos^{-1} 0.34$ **35.** $\sin^{-1} 0.75$

36. $\sin^{-1} (-0.4)$ **37.** $\cos^{-1} (-0.6)$ **38.** $\tan^{-1} (-0.2)$ **39.** $\tan^{-1} 2.25$

40. $\cos^{-1} (-0.8)$ **41.** $\sin^{-1} 0.99$ **42.** $\tan^{-1} 12$ **43.** $\cos^{-1} 0.55$

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 44–51.

SOLVING EQUATIONS **Solve the equation for θ. Round to three significant digits.**

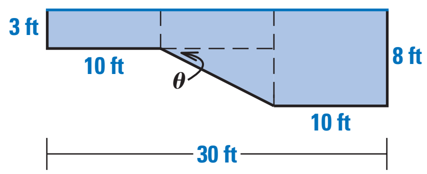
44. $\sin \theta = -0.35;\ 180° < \theta < 270°$ **45.** $\tan \theta = 2.4;\ 180° < \theta < 270°$

46. $\cos \theta = 0.43;\ 270° < \theta < 360°$ **47.** $\sin \theta = 0.8;\ 90° < \theta < 180°$

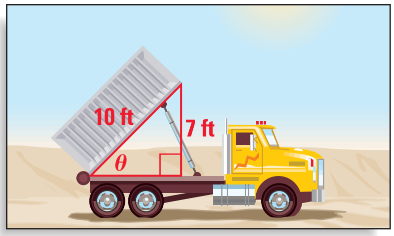
48. $\tan \theta = -2.1;\ 90° < \theta < 180°$ **49.** $\cos \theta = -0.72;\ 180° < \theta < 270°$

50. $\sin \theta = 0.2;\ 90° < \theta < 180°$ **51.** $\tan \theta = 0.9;\ 180° < \theta < 270°$

52. **SWIMMING POOL** The swimming pool shown in cross section at the right ranges in depth from 3 feet at the shallow end to 8 feet at the deep end. Find the angle of depression θ between the shallow end and the deep end.

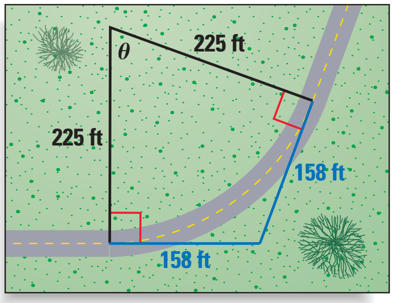


53. **DUMP TRUCK** The dump truck shown has a 10 foot bed. When tilted at its maximum angle, the bed reaches a height of 7 feet above its original position. What is the maximum angle θ that the truck bed can tilt?



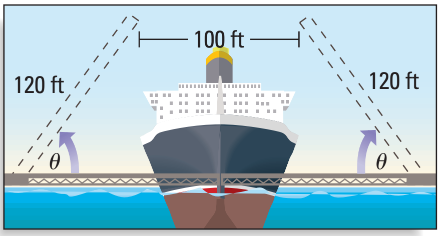
54. **GRANULAR ANGLE OF REPOSE** Look back at Example 4 on page 794. When whole corn is stored in a cone-shaped pile 20 feet high, the diameter of the pile's base is about 82 feet. Find the angle of repose for whole corn.

55. **ROAD DESIGN** Curves that connect two straight sections of a road are often constructed as arcs of circles. In the diagram, θ is the central angle of a circular arc that has a radius of 225 feet. Each radius line shown is perpendicular to one of the straight sections. The straight sections are therefore tangent to the arc. The extension of each straight section to their point of intersection is 158 feet in length. Find the degree measure of θ.

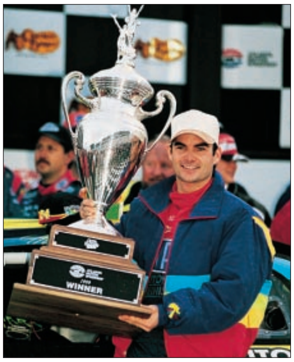


56. **DRAWBRIDGE** The Park Street Bridge in Alameda County, California, is a double-leaf drawbridge. Each leaf of the bridge is 120 feet long. A ship that is 100 feet wide needs to pass through the bridge. What is the minimum angle θ that each leaf of the bridge should be opened to in order to ensure that the ship will fit?

▶ Source: Alameda County Drawbridges

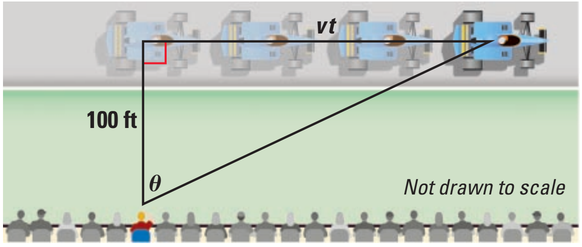


FOCUS ON PEOPLE



JEFF GORDON began racing in the top division of the National Association for Stock Car Auto Racing (NASCAR) in 1993. He has won three NASCAR division championships in the past four years.

57. **RACEWAY** Suppose you are at a raceway and are sitting on the straightaway, 100 feet from the center of the track. If a car traveling 145 miles per hour passes directly in front of you, at what angle do you have to turn your head to see the car t seconds later? Assume that the car is still on the straightaway and is traveling at a constant speed. (*Hint:* First convert 145 miles per hour to a speed v in feet per second. The expression vt represents the distance in feet traveled by the car.)



GEOMETRY CONNECTION **In Exercises 58–62, use the following information.** Consider a line with positive slope m that makes an angle θ with the x-axis (measuring counterclockwise from the x-axis).

58. Find the slope m of the line $y = 3x - 2$.

59. Find θ for the line $y = 3x - 2$.

60. **CRITICAL THINKING** How could you have found θ for the line $y = 3x - 2$ by using the slope m of the line? Write an equation relating θ and m.

61. Find an equation of the line that makes an angle of 58° with the x-axis and whose y-intercept is 3.

62. Find an equation of the line that makes an angle of 35° with the x-axis and whose x-intercept is 4.

Test Preparation

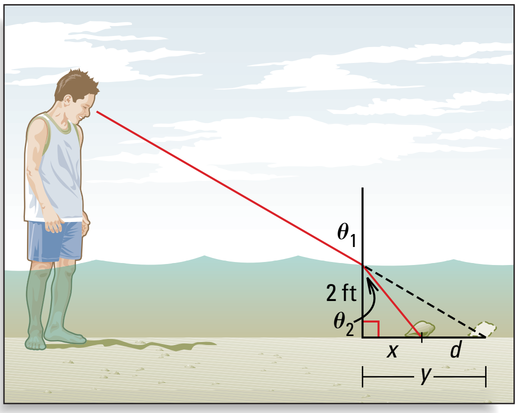
63. **MULTI-STEP PROBLEM** If you stand in shallow water and look at an object below the surface of the water, the object will look farther away from you than it really is. This is because when light rays pass between air and water, the water *refracts*, or bends, the light rays. The *index of refraction* for seawater is 1.341. This is the ratio of the sine of θ_1 to the sine of θ_2 for angles θ_1 and θ_2 below.

a. You are standing in seawater that is 2 feet deep and are looking at a shell at angle $\theta_1 = 60°$ (measured from a line perpendicular to the surface of the water). Find θ_2.

b. Find the distances x and y.

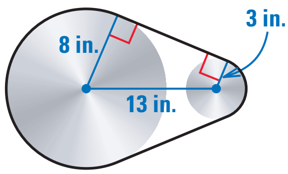
c. Find the distance d between where the shell is and where it appears to be.

d. *Writing* What happens to d as you move closer to the shell? Explain your reasoning.



★ Challenge

64. **LENGTH OF A PULLEY BELT** Find the length of the pulley belt shown at the right. (*Hint:* Partition the belt into four parts: the two straight segments, the arc around the small wheel, and the arc around the large wheel.)



MIXED REVIEW

SOLVING EQUATIONS **Solve the rational equation. Check for extraneous solutions.** (Review 9.6 for 13.5)

65. $\frac{6}{x} = \frac{7}{x+3}$ **66.** $\frac{3}{x-3} = \frac{7}{x}$ **67.** $\frac{-1}{4+x} = \frac{6}{2x}$

68. $\frac{3}{x+3} + 7 = \frac{-4}{x+3}$ **69.** $\frac{1}{x+2} = \frac{x}{2x+9}$ **70.** $\frac{3x}{x-2} = 2 + \frac{6}{x-2}$

CHOOSING NUMBERS **You have an equally likely chance of choosing any number 1 through 30. Find the probability of the given event.** (Review 12.3)

71. A multiple of 5 is chosen.

72. A prime number is chosen.

73. An even number is chosen.

74. A factor of 90 is chosen.

75. A number less than 12 is chosen.

76. A number greater than 23 is chosen.

EVALUATING FUNCTIONS **Use a calculator to evaluate the function. Round to four decimal places.** (Review 13.3 for 13.5)

77. $\sin 27°$ **78.** $\sin \frac{23\pi}{8}$ **79.** $\cos 67°$

80. $\sec \frac{53\pi}{9}$ **81.** $\tan 192°$ **82.** $\csc 219°$

QUIZ 2

Self-Test for Lessons 13.3 and 13.4

Use the given point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ. (Lesson 13.3)

1. $(-9, -16)$ **2.** $(7, -2)$ **3.** $(-1, 5)$ **4.** $(6, -11)$

5. $(3, 6)$ **6.** $(-12, 3)$ **7.** $(9, -5)$ **8.** $(-7, -8)$

Evaluate the function without using a calculator. (Lesson 13.3)

9. $\sin(-135°)$ **10.** $\tan \frac{8\pi}{3}$ **11.** $\cos(-420°)$ **12.** $\tan\left(-\frac{2\pi}{3}\right)$

13. $\sin \frac{5\pi}{3}$ **14.** $\cos 870°$ **15.** $\tan(-30°)$ **16.** $\sin \frac{23\pi}{6}$

Use a calculator to evaluate the expression in both radians and degrees. Round to three significant digits. (Lesson 13.4)

17. $\tan^{-1} 2.3$ **18.** $\sin^{-1}(-0.6)$ **19.** $\cos^{-1} 0.95$ **20.** $\sin^{-1} 0.23$

21. $\tan^{-1}(-4)$ **22.** $\cos^{-1}(-0.8)$ **23.** $\sin^{-1} 0.1$ **24.** $\tan^{-1} 10$

Solve the equation for θ. Round to three significant digits. (Lesson 13.4)

25. $\sin\theta = 0.25;\ 90° < \theta < 180°$

26. $\cos\theta = 0.21;\ 270° < \theta < 360°$

27. $\tan\theta = 7;\ 180° < \theta < 270°$

28. $\sin\theta = -0.44;\ 180° < \theta < 270°$

29. $\cos\theta = -0.3;\ 180° < \theta < 270°$

30. $\tan\theta = -4.5;\ 90° < \theta < 180°$

31. **LACROSSE** A lacrosse player throws a ball at an angle of 55° and at an initial speed of 40 feet per second. How far away should her teammate be to catch the ball at the same height from which it was thrown? (Lesson 13.3)