13.3 Trigonometric Functions of Any Angle

What you should learn

GOAL 1 Evaluate trigonometric functions of any angle.

GOAL 2 Use trigonometric functions to solve **real-life** problems, such as finding the distance a soccer ball is kicked in **Ex. 71**.

Why you should learn it

▼ To solve **real-life** problems, such as finding distances for a marching band on a football field in **Example 6**.

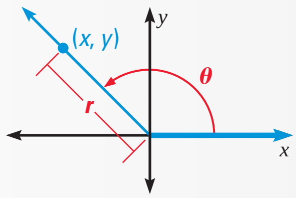
REAL LIFE

GOAL 1 EVALUATING TRIGONOMETRIC FUNCTIONS

In Lesson 13.1 you learned how to evaluate trigonometric functions of an acute angle. In this lesson you will learn to evaluate trigonometric functions of *any* angle.

GENERAL DEFINITION OF TRIGONOMETRIC FUNCTIONS

Let θ be an angle in standard position and (x, y) be any point (except the origin) on the terminal side of θ. The six trigonometric functions of θ are defined as follows.



$\sin\theta = \frac{y}{r}$ $\qquad$ $\csc\theta = \frac{r}{y},\ y \neq 0$

$\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{r}{x},\ x \neq 0$

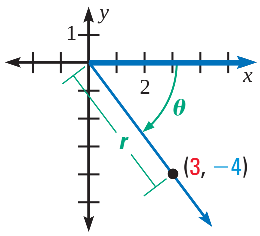
$\tan\theta = \frac{y}{x},\ x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y},\ y \neq 0$

Pythagorean theorem gives $r = \sqrt{x^2 + y^2}$.

For acute angles, these definitions give the same values as those given by the definitions in Lesson 13.1.

EXAMPLE 1 *Evaluating Trigonometric Functions Given a Point*

Let $(3, -4)$ be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.



SOLUTION

Use the Pythagorean theorem to find the value of r.

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{3^2 + (-4)^2}$$
$$= \sqrt{25}$$
$$= 5$$

Using $x = 3$, $y = -4$, and $r = 5$, you can write the following.

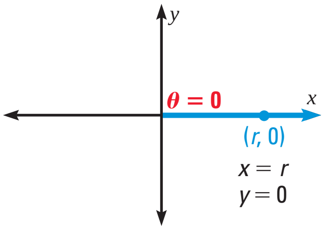
$\sin\theta = \frac{y}{r} = -\frac{4}{5}$ $\qquad$ $\csc\theta = \frac{r}{y} = -\frac{5}{4}$

$\cos\theta = \frac{x}{r} = \frac{3}{5}$ $\qquad$ $\sec\theta = \frac{r}{x} = \frac{5}{3}$

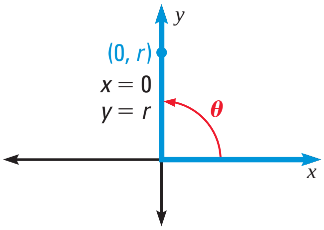
$\tan\theta = \frac{y}{x} = -\frac{4}{3}$ $\qquad$ $\cot\theta = \frac{x}{y} = -\frac{3}{4}$

If the terminal side of θ lies on an axis, then θ is a **quadrantal angle**. The diagrams below show the values of x and y for the quadrantal angles 0°, 90°, 180°, and 270°.

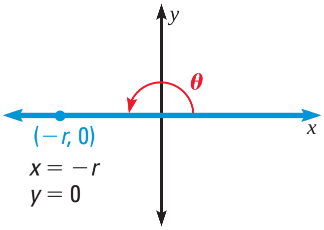
0° or 0 radians



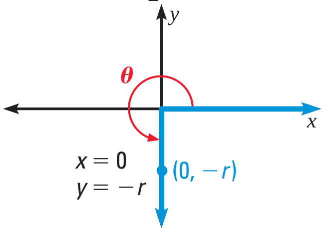
90° or $\frac{\pi}{2}$ radians



180° or π radians



270° or $\frac{3\pi}{2}$ radians



EXAMPLE 2 *Trigonometric Functions of a Quadrantal Angle*

Evaluate the six trigonometric functions of $\theta = 180°$.

Solution

When $\theta = 180°$, $x = -r$ and $y = 0$. The six trigonometric functions of θ are as follows.

$$\sin\theta = \frac{y}{r} = \frac{0}{r} = 0 \qquad \csc\theta = \frac{r}{y} = \frac{r}{0} = \text{undefined}$$

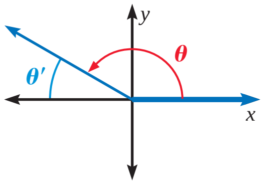
$$\cos\theta = \frac{x}{r} = \frac{-r}{r} = -1 \qquad \sec\theta = \frac{r}{x} = \frac{r}{-r} = -1$$

$$\tan\theta = \frac{y}{x} = \frac{0}{-r} = 0 \qquad \cot\theta = \frac{x}{y} = \frac{-r}{0} = \text{undefined}$$

....

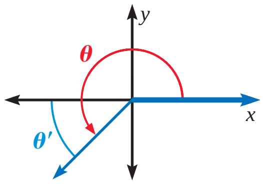
The values of trigonometric functions of angles greater than 90° (or less than 0°) can be found using corresponding acute angles called *reference angles*. Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' (read *theta prime*) formed by the terminal side of θ and the x-axis. The relationship between θ and θ' is given below for nonquadrantal angles θ such that $90° < \theta < 360°$ $\left(\frac{\pi}{2} < \theta < 2\pi\right)$.

$90° < \theta < 180°$; $\frac{\pi}{2} < \theta < \pi$



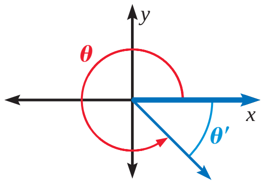
Degrees: $\theta' = 180° - \theta$
Radians: $\theta' = \pi - \theta$

$180° < \theta < 270°$; $\pi < \theta < \frac{3\pi}{2}$



Degrees: $\theta' = \theta - 180°$
Radians: $\theta' = \theta - \pi$

$270° < \theta < 360°$; $\frac{3\pi}{2} < \theta < 2\pi$



Degrees: $\theta' = 360° - \theta$
Radians: $\theta' = 2\pi - \theta$

EXAMPLE 3 *Finding Reference Angles*

Find the reference angle θ' for each angle θ.

a. $\theta = 320°$ **b.** $\theta = -\frac{5\pi}{6}$

SOLUTION

a. Because $270° < \theta < 360°$, the reference angle is $\theta' = 360° - 320° = 40°$.

b. Because θ is coterminal with $\frac{7\pi}{6}$ and $\pi < \frac{7\pi}{6} < \frac{3\pi}{2}$, the reference angle is $\theta' = \frac{7\pi}{6} - \pi = \frac{\pi}{6}$.

....

The signs of the trigonometric function values in the four quadrants can be determined from the function definitions. For instance, because $\cos\theta = \frac{x}{r}$ and r is always positive, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV.

CONCEPT SUMMARY — EVALUATING TRIGONOMETRIC FUNCTIONS

Use these steps to evaluate a trigonometric function of any angle θ.

1. Find the reference angle θ'.
2. Evaluate the trigonometric function for the angle θ'.
3. Use the quadrant in which θ lies to determine the sign of the trigonometric function value of θ. (See the diagram at the right.)

Signs of Function Values

Quadrant II	Quadrant I
$\sin\theta$, $\csc\theta$: +	$\sin\theta$, $\csc\theta$: +
$\cos\theta$, $\sec\theta$: −	$\cos\theta$, $\sec\theta$: +
$\tan\theta$, $\cot\theta$: −	$\tan\theta$, $\cot\theta$: +
Quadrant III	**Quadrant IV**
$\sin\theta$, $\csc\theta$: −	$\sin\theta$, $\csc\theta$: −
$\cos\theta$, $\sec\theta$: −	$\cos\theta$, $\sec\theta$: +
$\tan\theta$, $\cot\theta$: +	$\tan\theta$, $\cot\theta$: −

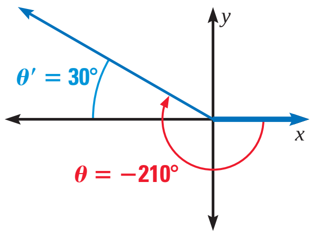
EXAMPLE 4 *Using Reference Angles to Evaluate Trigonometric Functions*

Evaluate (**a**) $\tan(-210°)$ and (**b**) $\csc\frac{11\pi}{4}$.

SOLUTION

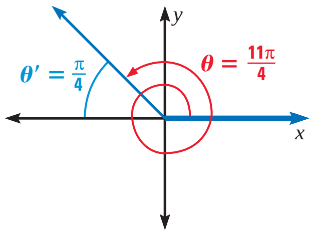
a. The angle $-210°$ is coterminal with $150°$. The reference angle is $\theta' = 180° - 150° = 30°$. The tangent function is negative in Quadrant II, so you can write:

$$\tan(-210°) = -\tan 30° = -\frac{\sqrt{3}}{3}$$



b. The angle $\frac{11\pi}{4}$ is coterminal with $\frac{3\pi}{4}$. The reference angle is $\theta' = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$. The cosecant function is positive in Quadrant II, so you can write:

$$\csc\frac{11\pi}{4} = \csc\frac{\pi}{4} = \sqrt{2}$$



STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

FOCUS ON APPLICATIONS

GOLF BALLS The dimples on a golf ball create pockets of air turbulence that keep the ball in the air for a longer period of time than if the ball were smooth. The longest drive of a golf ball on record is 473 yards, 2 feet, 6 inches.

DATA UPDATE
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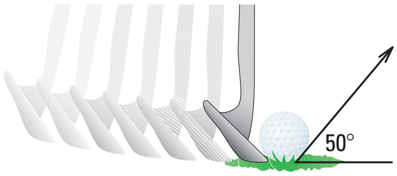
GOAL 2 USING TRIGONOMETRIC FUNCTIONS IN REAL LIFE

EXAMPLE 5 *Calculating Projectile Distance*

GOLF The horizontal distance d (in feet) traveled by a projectile with an initial speed v (in feet per second) is given by

$$d = \frac{v^2}{32}\sin 2\theta$$



where θ is the angle at which the projectile is launched. Estimate the horizontal distance traveled by a golf ball that is hit at an angle of 50° with an initial speed of 105 feet per second. (This model neglects air resistance and wind conditions. It also assumes that the projectile's starting and ending heights are the same.)

SOLUTION

The horizontal distance given by the model is:

$$d = \frac{v^2}{32}\sin 2\theta$$ Write distance model.

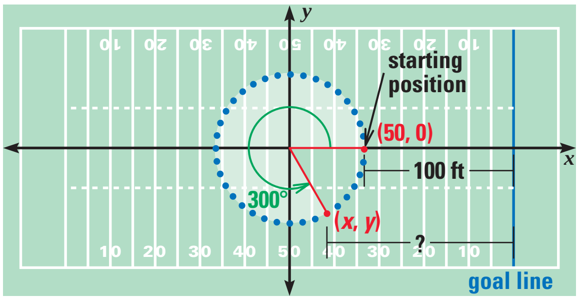
$$= \frac{105^2}{32}\sin(2 \cdot 50^\circ) \approx 339 \text{ feet}$$ Substitute and use a calculator.

▶ The golf ball travels a horizontal distance of about 339 feet.

REAL LIFE *Marching Band*

EXAMPLE 6 *Modeling with Trigonometric Functions*

Your school's marching band is performing at halftime during a football game. In the last formation, the band members form a circle 100 feet wide in the center of the field. Your starting position is 100 feet from the goal line, where you will exit the field. How far from the goal line will you be after you have marched 300° around the circle?



SOLUTION

The radius of the circle is $r = 50$. So, you can write:

$$\cos 300^\circ = \frac{x}{r}$$ Use definition of cosine.

$$\frac{1}{2} = \frac{x}{50}$$ Substitute.

$$25 = x$$ Solve for x.

▶ You will be $100 + (50 - 25) = 125$ feet from the goal line.

GUIDED PRACTICE

Vocabulary Check ✓

1. Define the terms quadrantal angle and reference angle.

Concept Check ✓

2. Given an angle θ in Quadrant III, explain how you can use a reference angle to find $\sin\theta$.
3. Explain why $\tan 270°$ is undefined.
4. In which quadrant(s) must θ lie for $\cos\theta$ to be positive?

Skill Check ✓

5. Let $(-4, -5)$ be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.

Sketch the angle. Then find its reference angle.

6. $\frac{7\pi}{4}$
7. $-120°$
8. $\frac{7\pi}{8}$
9. $390°$
10. $-\frac{2\pi}{3}$
11. $-370°$
12. $\frac{2\pi}{3}$
13. $230°$

Evaluate the function without using a calculator.

14. $\cos\left(-\frac{4\pi}{3}\right)$
15. $\tan 240°$
16. $\sin\frac{7\pi}{4}$
17. $\csc(-225°)$
18. $\cot\left(-\frac{3\pi}{4}\right)$
19. $\cos 240°$
20. $\sec\frac{11\pi}{6}$
21. $\tan\frac{5\pi}{6}$
22. **MARCHING BAND** Look back at Example 6 on page 787. Suppose you marched 135° around the circle from the same starting position. How far from the goal line would you be?

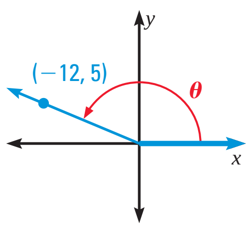
PRACTICE AND APPLICATIONS

STUDENT HELP

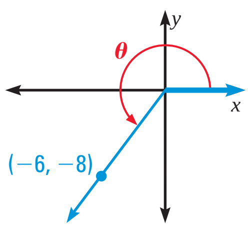
Extra Practice to help you master skills is on p. 958.

USING A POINT **Use the given point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.**

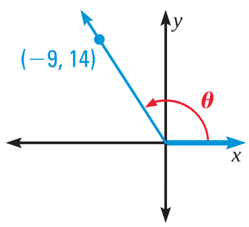
23.



24.



25.



26. $(-12, -15)$
27. $(-1, 1)$
28. $(15, -8)$
29. $(6, -9)$
30. $(7, 10)$
31. $(1, -\sqrt{3})$
32. $(-3, -4)$
33. $(-15, 5\sqrt{7})$

QUADRANTAL ANGLES **Evaluate the six trigonometric functions of θ.**

34. $\theta = 90°$
35. $\theta = 270°$
36. $\theta = 0°$

FINDING REFERENCE ANGLES **Sketch the angle. Then find its reference angle.**

37. $240°$
38. $-515°$
39. $-170°$
40. $315°$
41. $-440°$
42. $-\frac{3\pi}{4}$
43. $\frac{25\pi}{4}$
44. $-\frac{11\pi}{3}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 23–33
Example 2: Exs. 34–36
Example 3: Exs. 37–44
Example 4: Exs. 45–60
Example 5: Exs. 69–71
Example 6: Exs. 72–76

EVALUATING FUNCTIONS **Evaluate the function without using a calculator.**

45. $\cos 315°$ **46.** $\cos(-210°)$ **47.** $\csc(-240°)$ **48.** $\tan 210°$

49. $\sec 780°$ **50.** $\sin 225°$ **51.** $\cos(-225°)$ **52.** $\tan(-120°)$

53. $\cot \frac{11\pi}{6}$ **54.** $\sec \frac{9\pi}{4}$ **55.** $\sin\left(-\frac{5\pi}{6}\right)$ **56.** $\cos \frac{5\pi}{3}$

57. $\sin\left(-\frac{17\pi}{6}\right)$ **58.** $\sec \frac{23\pi}{6}$ **59.** $\csc \frac{17\pi}{3}$ **60.** $\cot\left(-\frac{13\pi}{4}\right)$

STUDENT HELP

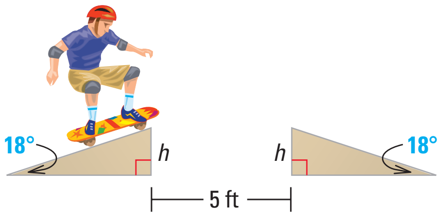
Study Tip
Make sure your calculator is in radian mode when finding trigonometric functions of angles measured in radians.

USING A CALCULATOR **Use a calculator to evaluate the function. Round the result to four decimal places.**

61. $\sec 137°$ **62.** $\cot 400°$ **63.** $\sin(-10°)$ **64.** $\csc 540°$

65. $\cot\left(-\frac{4\pi}{5}\right)$ **66.** $\sec \frac{11\pi}{2}$ **67.** $\cos \frac{6\pi}{5}$ **68.** $\csc \frac{23\pi}{8}$

69. **SKATEBOARDING** A skateboarder is setting up two ramps for a jump as shown. He wants to jump off one ramp and land on the other. If the ramps are placed 5 feet apart, at what speed must the skateboarder launch off the first ramp to land on the second ramp?



70. **VOLLEYBALL** While playing a game of volleyball, you set the ball to your teammate. You hit the ball with an initial speed of 24 feet per second at an angle of 70°. About how far away should your teammate be to receive your set?

71. **SOCCER** You and a friend are playing soccer. Both of you kick the ball with an initial speed of 42 feet per second. Your kick was projected at an angle of 45° and your friend's kick was projected at an angle of 60°. About how much farther will your soccer ball go than your friend's soccer ball?

FOCUS ON CAREERS

CARTOGRAPHER
Cartographers compile information from aerial photographs and satellite data to map Earth's surface. A map's circles of latitude and longitude, as discussed in Exs. 73 and 74, are used to describe location.

CAREER LINK
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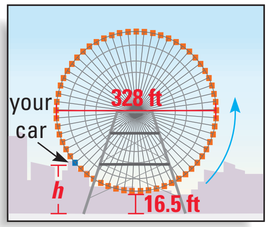
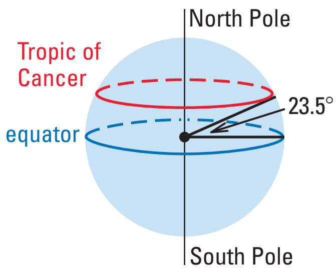
72. **FERRIS WHEEL** The largest Ferris wheel in operation is the Cosmolock 21 at Yokohama City, Japan. It has a diameter of 328 feet. Passengers board the cars at the bottom of the wheel, about 16.5 feet above the ground. Imagine that you have boarded the Cosmolock 21. The wheel rotates 312° and then stops. How high above the ground are you?



SOCIAL STUDIES CONNECTION **In Exercises 73 and 74, use the information below.**
The Tropic of Cancer is the circle of latitude farthest north of the equator where the sun can appear directly overhead. It lies 23.5° north of the equator, as shown below.

73. Find the circumference of the Tropic of Cancer using 3960 miles as Earth's approximate radius.

74. What is the distance between two points that lie directly across from each other (through the axis) on the Tropic of Cancer?



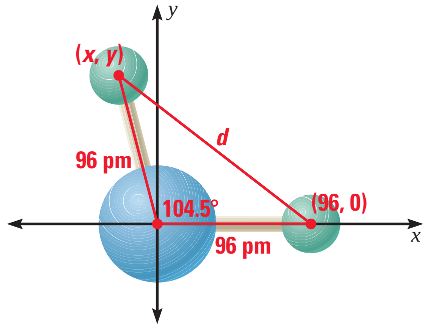
STUDENT HELP

Look Back
For help with the distance formula, see p. 589.

SCIENCE CONNECTION **In Exercises 75 and 76, use the following information.** When two atoms in a molecule are bonded to a common atom, chemists are interested in both the bond angle and the bond length. A water molecule (H_2O) is made up of two hydrogen atoms bonded to an oxygen atom. The diagram below shows a coordinate plane superimposed on a cross section of a water molecule.

75. In the diagram, coordinates are given in picometers (pm). (Note: 1 pm $= 10^{-12}$ m). If the center of one hydrogen atom has coordinates (96, 0), find the coordinates (x, y) of the center of the other hydrogen atom.

76. Use your answer to Exercise 75 and the distance formula to find the distance d (in picometers) between the centers of the two hydrogen atoms.



Test Preparation

77. MULTIPLE CHOICE What is the value of $\sec\left(\frac{40\pi}{3}\right)$?

Ⓐ -2 Ⓑ $-\sqrt{2}$ Ⓒ $-\frac{\sqrt{2}}{2}$ Ⓓ $-\frac{1}{2}$ Ⓔ $\sqrt{2}$

78. MULTIPLE CHOICE What is the approximate horizontal distance traveled by a football that is kicked at an angle of 40° with an initial speed of 70 feet per second?

Ⓐ 98 feet Ⓑ 142 feet Ⓒ 151 feet Ⓓ 157 feet Ⓔ 280 feet

★ Challenge

79. CRITICAL THINKING If θ is an angle in Quadrant II and $\tan\theta = -2$, find the values of the other five trigonometric functions of θ.

80. CRITICAL THINKING If θ is an angle in Quadrant III and $\cos\theta = -0.64$, find the values of the other five trigonometric functions of θ.

MIXED REVIEW

HORIZONTAL LINE TEST **Graph the function. Then use the graph to determine whether the inverse of f is a function.** (Review 7.4 for 13.4)

81. $f(x) = x - 3$ **82.** $f(x) = 4x + 5$ **83.** $f(x) = 5x^2$

84. $f(x) = 5x^3$ **85.** $f(x) = 3x^2 - 7$ **86.** $f(x) = -|x + 2|$

CHOOSING CARDS **A card is randomly drawn from a standard 52-card deck. Find the probability of the given event. (A face card is a king, queen, or jack.)** (Review 12.4)

87. a king and a diamond **88.** a jack or a club **89.** a ten or a face card

SOLVING TRIANGLES **Solve $\triangle ABC$ using the diagram and the given measurements.** (Review 13.1)

90. $A = 62°, b = 5$ **91.** $B = 20°, c = 22$

92. $B = 31°, a = 17$ **93.** $A = 50°, c = 3$

94. $B = 75°, b = 34$ **95.** $A = 83°, a = 50$