13.2 General Angles and Radian Measure

What *you should learn*

GOAL 1 Measure angles in standard position using degree measure and radian measure.

GOAL 2 Calculate arc lengths and areas of sectors, as applied in **Example 6**.

Why *you should learn it*

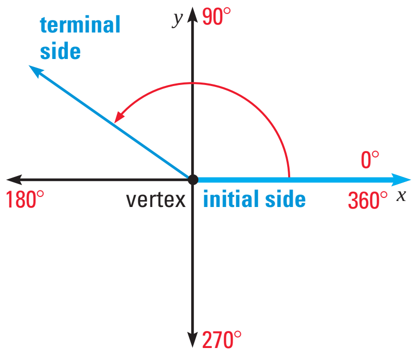
▼ To solve **real-life** problems, such as finding the angle generated by a rotating figure skater in **Exs. 77–79**.

REAL LIFE

GOAL 1 ANGLES IN STANDARD POSITION

In Lesson 13.1 you worked only with acute angles (angles measuring between 0° and 90°). In this lesson you will study angles whose measures can be any real numbers.

Recall that an angle is formed by two rays that have a common endpoint, called the vertex. You can generate any angle by fixing one ray, called the **initial side**, and rotating the other ray, called the **terminal side**, about the vertex. In a coordinate plane, an angle whose vertex is at the origin and whose initial side is the positive x-axis is in **standard position**.



The measure of an angle is determined by the amount and direction of rotation from the initial side to the terminal side. The angle measure is positive if the rotation is counterclockwise, and negative if the rotation is clockwise. The terminal side of an angle can make more than one complete rotation.

EXAMPLE 1 *Drawing Angles in Standard Position*

Draw an angle with the given measure in standard position. Then tell in which quadrant the terminal side lies.

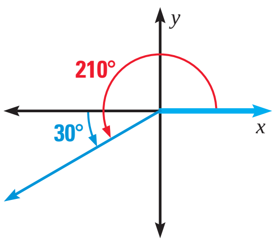
a. 210° **b.** −45° **c.** 510°

SOLUTION

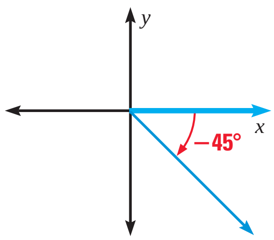
a. Use the fact that 210° = 180° + 30°. So, the terminal side is 30° counterclockwise past the negative x-axis.

b. Because −45° is negative, the terminal side is 45° clockwise from the positive x-axis.

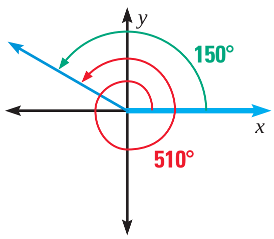
c. Use the fact that 510° = 360° + 150°. So, the terminal side makes one complete revolution counterclockwise and continues another 150°.



Terminal side in Quadrant III



Terminal side in Quadrant IV



Terminal side in Quadrant II

In Example 1 the angles 510° and 150° are *coterminal*. Two angles in standard position are **coterminal** if their terminal sides coincide. An angle coterminal with a given angle can be found by adding or subtracting multiples of 360°.

EXAMPLE 2 *Finding Coterminal Angles*

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

Find one positive angle and one negative angle that are coterminal with (**a**) −60° and (**b**) 495°.

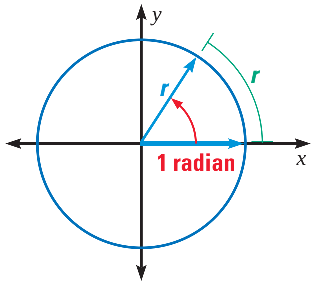
Solution

There are many such angles, depending on what multiple of 360° is added or subtracted.

a. Positive coterminal angle: $-60° + 360° = 300°$
Negative coterminal angle: $-60° - 360° = -420°$

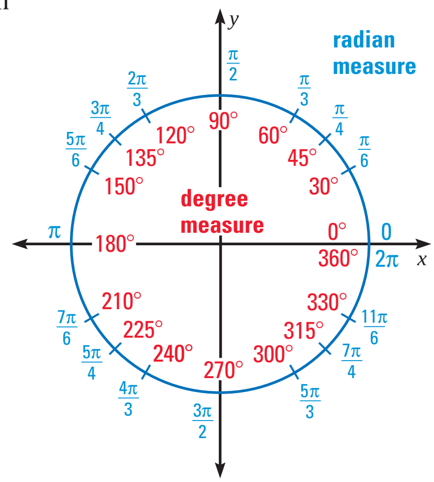
b. Positive coterminal angle: $495° - 360° = 135°$
Negative coterminal angle: $495° - 2(360°) = -225°$

So far, all the angles you have worked with have been measured in degrees. You can also measure angles in *radians*. To define a radian, consider a circle with radius r centered at the origin. One **radian** is the measure of an angle in standard position whose terminal side intercepts an arc of length r.



Because the circumference of a circle is $2\pi r$, there are 2π radians in a full circle. Degree measure and radian measure are therefore related by the equation $360° = 2\pi$ radians, or $180° = \pi$ radians.

The diagram shows equivalent radian and degree measures for special angles from 0° to 360° (0 radians to 2π radians).



STUDENT HELP

Study Tip
When no units of angle measure are specified, radian measure is implied. For instance, $\theta = 2$ means that $\theta = 2$ radians.

You may find it helpful to memorize the equivalent degree and radian measures of special angles in the first quadrant and for $90° = \frac{\pi}{2}$ radians. All other special angles are just multiples of these angles.

You can use the following rules to convert degrees to radians and radians to degrees.

CONVERSIONS BETWEEN DEGREES AND RADIANS

- To rewrite a degree measure in radians, multiply by $\frac{\pi \text{ radians}}{180°}$.
- To rewrite a radian measure in degrees, multiply by $\frac{180°}{\pi \text{ radians}}$.

EXAMPLE 3 *Converting Between Degrees and Radians*

a. Convert 110° to radians. **b.** Convert $-\frac{\pi}{9}$ radians to degrees.

SOLUTION

a. $110° = 110°\left(\frac{\pi \text{ radians}}{180°}\right)$

$= \frac{11\pi}{18}$ radians

✓**CHECK** Check that your answer is reasonable:

The angle 110° is between the special angles 90° and 120°. The angle $\frac{11\pi}{18}$ is between the same special angles: $\frac{\pi}{2} = \frac{9\pi}{18}$ and $\frac{2\pi}{3} = \frac{12\pi}{18}$.

b. $-\frac{\pi}{9} = \left(-\frac{\pi}{9} \text{ radians}\right)\left(\frac{180°}{\pi \text{ radians}}\right)$

$= -20°$

✓**CHECK** Check that your answer is reasonable:

The angle $-\frac{\pi}{9}$ is between the special angles 0 and $-\frac{\pi}{6}$. The angle $-20°$ is between the same special angles: 0° and $-30°$.

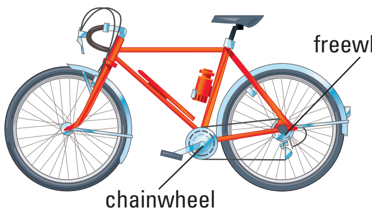
REAL LIFE
Bicycles

EXAMPLE 4 *Measuring an Angle for a Bicycle*

A bicycle's *gear ratio* is the number of times the freewheel turns for every one turn of the chainwheel. The table shows the number of teeth in the freewheel and chainwheel for the first 5 gears on an 18-speed touring bicycle. In fourth gear, if the chainwheel completes 3 rotations, through what angle does the freewheel turn? Give your answer in both degrees and radians. ▶ Source: *The All New Complete Book of Bicycling*

STUDENT HELP
INTERNET
APPLICATION LINK
Visit our Web site www.mcdougallittell.com for more information about bicycle gears in Example 4.

Gear number	Number of teeth in freewheel	Number of teeth in chainwheel
1	32	24
2	26	24
3	22	24
4	32	40
5	19	24



SOLUTION

In fourth gear, the gear ratio is $\frac{40}{32}$. For every one turn of the chainwheel in this gear, the freewheel makes 1.25 rotations. The measure of the angle θ through which the freewheel turns when the chainwheel completes 3 rotations is:

$$\theta = (3.75 \text{ rotations})\left(\frac{360°}{1 \text{ rotation}}\right) = 1350°$$

To find the angle measure in radians, multiply by $\frac{\pi \text{ radians}}{180°}$:

$$\theta = 1350°\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{15\pi}{2} \text{ radians}$$

GOAL 2 ARC LENGTHS AND AREAS OF SECTORS

A **sector** is a region of a circle that is bounded by two radii and an arc of the circle. The **central angle** θ of a sector is the angle formed by the two radii. There are simple formulas for the arc length and area of a sector when the central angle is measured in radians.

STUDENT HELP

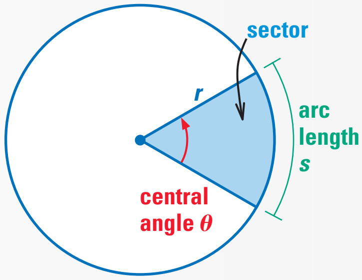
Derivations
For a derivation of the formula for arc length, see p. 900.

ARC LENGTH AND AREA OF A SECTOR

The arc length s and area A of a sector with radius r and central angle θ (measured in radians) are as follows.

Arc length: $s = r\theta$

Area: $A = \frac{1}{2}r^2\theta$



EXAMPLE 5 *Finding Arc Length and Area*

Find the arc length and area of a sector with a radius of 9 cm and a central angle of 60°.

SOLUTION

First convert the angle measure to radians.

$$\theta = 60°\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{3} \text{ radians}$$

Then use the formulas for arc length and area.

$$\text{Arc length: } s = r\theta = 9\left(\frac{\pi}{3}\right) = 3\pi \text{ centimeters}$$

$$\text{Area: } A = \frac{1}{2}r^2\theta = \frac{1}{2}(9^2)\left(\frac{\pi}{3}\right) = \frac{27\pi}{2} \text{ square centimeters}$$

FOCUS ON APPLICATIONS

REAL LIFE **SPACE NEEDLE** The restaurant at the top of the Space Needle in Seattle, Washington, is circular and has a radius of 47.25 feet. The dining part of the restaurant (by the windows) revolves, making about one complete revolution per hour.

EXAMPLE 6 *Finding an Angle and Arc Length*

SPACE NEEDLE Read the photo caption at the left. You go to dinner at the Space Needle and sit at a window table at 6:42 P.M. Your dinner ends at 8:18 P.M. Through what angle do you rotate during your stay? How many feet do you revolve?

SOLUTION

You spend 96 minutes at dinner. Because the Space Needle makes one complete revolution every 60 minutes, your angle of rotation is:

$$\theta = \frac{96}{60}(2\pi) = \frac{16\pi}{5} \text{ radians}$$

Because the radius is 47.25 feet, you move through an arc length of:

$$s = r\theta = 47.25\left(\frac{16\pi}{5}\right) \approx 475 \text{ feet}$$

GUIDED PRACTICE

Vocabulary Check ✓

1. In your own words, describe what a radian is.

Concept Check ✓

2. ERROR ANALYSIS An error has been made in finding the area of a sector with a radius of 5 inches and a central angle of 25°. Find and correct the error.

$$\cancel{A = \frac{1}{2}(5^2)(25) = 312.5 \text{ in.}^2}$$

3. How does the sign of an angle's measure determine its direction of rotation?

4. A circle has radius r. What is the length of the arc corresponding to a central angle of π radians?

Skill Check ✓

Draw an angle with the given measure in standard position. Then find one positive and one negative coterminal angle.

5. 60° **6.** $-45°$ **7.** $\frac{7\pi}{4}$ **8.** 300°

9. $-\frac{3\pi}{2}$ **10.** $\frac{7\pi}{8}$ **11.** 150° **12.** $-\frac{5\pi}{4}$

Rewrite each degree measure in radians and each radian measure in degrees.

13. 30° **14.** 100° **15.** 260° **16.** $-320°$

17. $\frac{7\pi}{4}$ **18.** $\frac{18\pi}{4}$ **19.** $\frac{\pi}{12}$ **20.** $-\frac{5\pi}{2}$

Find the arc length and area of a sector with the given radius r and central angle θ.

21. $r = 4$ in., $\theta = 55°$ **22.** $r = 5$ m, $\theta = 135°$ **23.** $r = 2$ cm, $\theta = 85°$

24. SPACE NEEDLE Recall from Example 6 on page 779 that the circular restaurant at the Space Needle has a radius of 47.25 feet and rotates about once per hour. If you are seated at a window table from 6:00 P.M. to 8:10 P.M., through what angle do you rotate? How many feet do you revolve?

PRACTICE AND APPLICATIONS

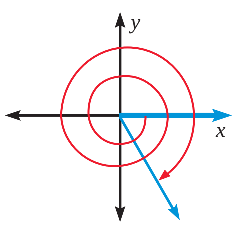
STUDENT HELP

Extra Practice to help you master skills is on p. 958.

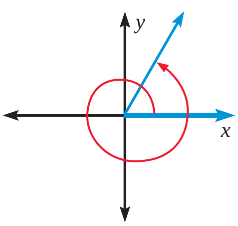
VISUAL THINKING **Match the angle measure with the angle.**

25. $-210°$ **26.** 420° **27.** $-\frac{13\pi}{3}$

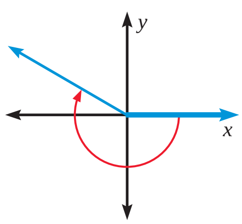
A.



B.



C.



DRAWING ANGLES **Draw an angle with the given measure in standard position.**

28. 144° **29.** $\frac{2\pi}{9}$ **30.** $-15°$ **31.** $-\frac{7\pi}{6}$

32. $\frac{19\pi}{12}$ **33.** 1620° **34.** -5π **35.** $-\frac{13\pi}{4}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 25–35
Example 2: Exs. 36–43
Example 3: Exs. 44–59
Example 4: Exs. 77–81
Example 5: Exs. 60–68
Example 6: Exs. 82–88

FINDING COTERMINAL ANGLES **Find one positive angle and one negative angle coterminal with the given angle.**

36. $55°$ **37.** $210°$ **38.** $420°$ **39.** $780°$

40. $\frac{13\pi}{2}$ **41.** $\frac{17\pi}{4}$ **42.** $\frac{24\pi}{7}$ **43.** $\frac{16\pi}{3}$

CONVERTING MEASURES **Rewrite each degree measure in radians and each radian measure in degrees.**

44. $25°$ **45.** $225°$ **46.** $160°$ **47.** $45°$

48. $-110°$ **49.** $325°$ **50.** $400°$ **51.** $-290°$

52. $\frac{7\pi}{3}$ **53.** $-\frac{9\pi}{2}$ **54.** $\frac{\pi}{10}$ **55.** $-\frac{5\pi}{12}$

56. $\frac{7\pi}{15}$ **57.** $-\frac{15\pi}{4}$ **58.** $-\frac{5\pi}{6}$ **59.** $\frac{8\pi}{5}$

FINDING ARC LENGTH AND AREA **Find the arc length and area of a sector with the given radius *r* and central angle *θ*.**

60. $r = 3$ in., $\theta = \frac{\pi}{4}$ **61.** $r = 3$ ft, $\theta = \frac{\pi}{18}$ **62.** $r = 2$ cm, $\theta = \frac{9\pi}{20}$

63. $r = 12$ in., $\theta = 90°$ **64.** $r = 5$ m, $\theta = 120°$ **65.** $r = 15$ mm, $\theta = 175°$

66. $r = 4$ ft, $\theta = 200°$ **67.** $r = 16$ cm, $\theta = 50°$ **68.** $r = 20$ ft, $\theta = 270°$

EVALUATING FUNCTIONS **Evaluate the trigonometric function using a calculator if necessary. If possible, give an exact answer.**

69. $\sin \frac{\pi}{6}$ **70.** $\cos \frac{\pi}{4}$ **71.** $\tan \frac{\pi}{3}$ **72.** $\cos \frac{4\pi}{11}$

73. $\cot \frac{\pi}{5}$ **74.** $\sec \frac{\pi}{8}$ **75.** $\sin \frac{2\pi}{9}$ **76.** $\csc \frac{3\pi}{10}$

FIGURE SKATING **In Exercises 77–79, use the following information.** The number of revolutions made by a figure skater for each type of Axel jump is given. Determine the measure of the angle generated as the skater performs the jump. Give the answer in both degrees and radians.

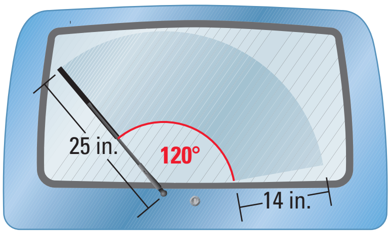
77. Single Axel: $1\frac{1}{2}$ **78.** Double Axel: $2\frac{1}{2}$ **79.** Triple Axel: $3\frac{1}{2}$

80. **TIME IN SCHOOL** You are in school from 8:00 A.M. to 3:00 P.M. Draw a diagram that shows the number of rotations completed by the minute hand of a clock during this time. Find the measure of the angle generated by the minute hand. Give the answer in both degrees and radians.

81. **BICYCLE GEARS** Look back at Example 4 on page 778. In fifth gear, if the bicycle's chainwheel completes 4 rotations, through what angle does the freewheel turn? Give your answer in both degrees and radians.

82. **FARMING TECHNOLOGY** A sprinkler system on a farm rotates 140° and sprays water up to 35 meters. Draw a diagram that shows the region that can be irrigated with the sprinkler. Then find the area of the region.

83. **WINDSHIELD WIPERS** A car's rear windshield wiper rotates 120° as shown. Find the area covered by the wiper.



FOCUS ON PEOPLE

AXEL PAULSEN, a Norwegian speed skating champion, invented the Axel jump in 1882. It is the only jump in figure skating that requires taking off from a forward position.

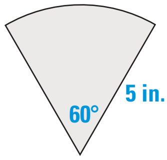
SPIRAL STAIRS **In Exercises 84–86, use the following information.**
A spiral staircase has 13 steps. Each step is a sector with a radius of 36 inches and a central angle of $\frac{\pi}{7}$.

84. What is the length of the arc formed by the outer edge of each step?

85. Through what angle would you rotate by climbing the stairs? Include a fourteenth turn for stepping up on the landing.

86. How many square inches of carpeting would you need to cover the 13 steps?

SNOW CONES **In Exercises 87 and 88, use the following information.**
You are starting a business selling homemade snow cones in paper cups. You cut out a paper cup in the shape of a sector.



87. The sector has a central angle of 60° and a radius of 5 inches. When you shape the sector into a cone without overlapping edges, what will the cone's diameter be?

88. Suppose you want to make a cone that has a diameter of 4 inches and a slant height of 6 inches. What should the radius and central angle of the sector be?

Test Preparation

QUANTITATIVE COMPARISON **In Exercises 89 and 90, choose the statement that is true about the given quantities.**

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

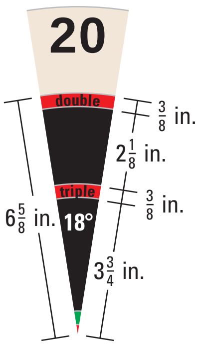
Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
89.	Arc length of a sector with $r = 2$ inches and $\theta = 45°$	Arc length of a sector with $r = 2.5$ inches and $\theta = \frac{\pi}{5}$
90.	Area of a sector with $r = 2$ inches and $\theta = 45°$	Area of a sector with $r = 2.5$ inches and $\theta = \frac{\pi}{5}$

★ Challenge

DARTS **In Exercises 91 and 92, use the following information.**
A dart board is divided into 20 sectors. Each sector is worth a point value from 1 to 20 and has shaded regions that double or triple this value. The 20 point sector is shown at the right.



91. Find the area of the sector. Then find the areas of the double region and the triple region in the sector.

92. If you throw a dart and it randomly lands somewhere inside the sector, what is the probability that it lands within the double region? within the triple region?

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

PROPERTIES OF SQUARE ROOTS **Simplify the expression.** (Review 5.3 for 13.3)

93. $\sqrt{275}$ **94.** $\sqrt{1216}$ **95.** $\sqrt{8} \cdot \sqrt{32}$ **96.** $\sqrt{18} \cdot \sqrt{24}$

97. $\sqrt{\frac{7}{16}}$ **98.** $\sqrt{\frac{11}{36}}$ **99.** $\frac{\sqrt{8}}{\sqrt{7}}$ **100.** $\frac{\sqrt{12}}{\sqrt{5}}$

EVALUATING EXPRESSIONS **Evaluate the expression $\frac{x^2}{2y + 5}$ for the given values of x and y.** (Review 1.2 for 13.3)

101. $x = 6, y = 11$ **102.** $x = 3, y = -3$ **103.** $x = 12, y = 15$

104. $x = -1, y = -5$ **105.** $x = -10, y = 16$ **106.** $x = -20, y = -25$

WRITING EQUATIONS **Write the standard form of the equation of the parabola with the given focus and vertex at (0, 0).** (Review 10.2)

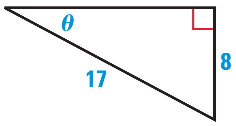
107. $(5, 0)$ **108.** $(-3, 0)$ **109.** $(6, 0)$

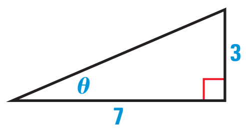
110. $(0, -12)$ **111.** $(0, -4.4)$ **112.** $(0, 15)$

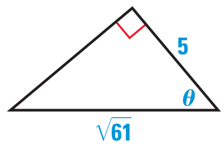
QUIZ 1

Self-Test for Lessons 13.1 and 13.2

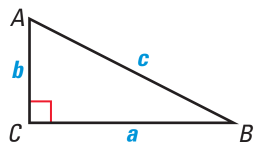
Evaluate the six trigonometric functions of θ. (Lesson 13.1)

1. 

2. 

3. 

Solve △*ABC* using the diagram at the right and the given measurements. (Lesson 13.1)



4. $B = 50°, a = 18$ **5.** $A = 33°, c = 12$

6. $A = 10°, a = 3$ **7.** $B = 71°, c = 2.3$

Find one positive angle and one negative angle coterminal with the given angle. (Lesson 13.2)

8. $25°$ **9.** $-\frac{14\pi}{3}$ **10.** $\frac{33\pi}{4}$ **11.** $-6200°$

Find the arc length and area of a sector with the given radius r and central angle θ. (Lesson 13.2)

12. $r = 6$ m, $\theta = \frac{\pi}{3}$ **13.** $r = 2$ ft, $\theta = \frac{5\pi}{6}$ **14.** $r = 8$ cm, $\theta = 20°$

15. $r = 22$ in., $\theta = 220°$ **16.** $r = 5$ ft, $\theta = 75°$ **17.** $r = 12$ mm, $\theta = 160°$

18. **THE BEST DEAL** Decide which of the two pizza slices shown is the best deal. Explain your reasoning. (Lesson 13.2)