9.3

Graphing General Rational Functions

What you should learn

GOAL 1 Graph general rational functions.

GOAL 2 Use the graph of a rational function to solve **real-life** problems, such as determining the efficiency of packaging in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the energy expenditure of a parakeet in **Ex. 39**.

REAL LIFE

GOAL 1 GRAPHING RATIONAL FUNCTIONS

In Lesson 9.2 you learned how to graph rational functions of the form

$$f(x) = \frac{p(x)}{q(x)}$$

for which $p(x)$ and $q(x)$ are linear polynomials and $q(x) \neq 0$. In this lesson you will learn how to graph rational functions for which $p(x)$ and $q(x)$ may be higher-degree polynomials.

CONCEPT SUMMARY — **GRAPHS OF RATIONAL FUNCTIONS**

Let $p(x)$ and $q(x)$ be polynomials with no common factors other than 1. The graph of the rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$$

has the following characteristics.

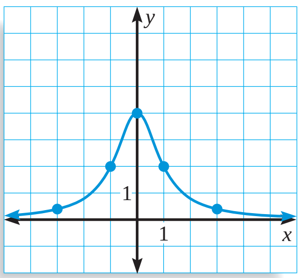
1. The x-intercepts of the graph of f are the real zeros of $p(x)$.
2. The graph of f has a vertical asymptote at each real zero of $q(x)$.
3. The graph of f has at most one horizontal asymptote.
   - If $m < n$, the line $y = 0$ is a horizontal asymptote.
   - If $m = n$, the line $y = \frac{a_m}{b_n}$ is a horizontal asymptote.
   - If $m > n$, the graph has no horizontal asymptote. The graph's end behavior is the same as the graph of $y = \frac{a_m}{b_n}x^{m-n}$.

EXAMPLE 1 *Graphing a Rational Function ($m < n$)*

Graph $y = \frac{4}{x^2 + 1}$. State the domain and range.

SOLUTION

The numerator has no zeros, so there is no x-intercept. The denominator has no real zeros, so there is no vertical asymptote. The degree of the numerator (0) is less than the degree of the denominator (2), so the line $y = 0$ (the x-axis) is a horizontal asymptote. The *bell-shaped* graph passes through the points $(-3, 0.4)$, $(-1, 2)$, $(0, 4)$, $(1, 2)$, and $(3, 0.4)$. The domain is all real numbers, and the range is $0 < y \leq 4$.



EXAMPLE 2 *Graphing a Rational Function ($m = n$)*

Graph $y = \dfrac{3x^2}{x^2 - 4}$.

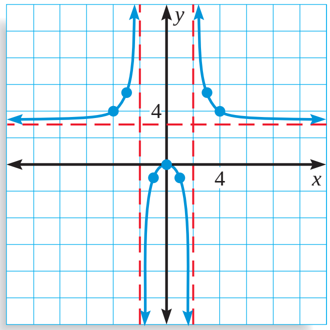
STUDENT HELP

Look Back
For help with finding zeros, see p. 259.

SOLUTION

The numerator has 0 as its only zero, so the graph has one x-intercept at $(0, 0)$. The denominator can be factored as $(x + 2)(x - 2)$, so the denominator has zeros -2 and 2. This implies that the lines $x = -2$ and $x = 2$ are vertical asymptotes of the graph. The degree of the numerator (2) is equal to the degree of the denominator (2), so the horizontal asymptote is $y = \dfrac{a_m}{b_n} = 3$. To draw the graph, plot points between and beyond the vertical asymptotes.

	x	y
To the left of $x = -2$	-4	4
	-3	5.4
Between $x = -2$ and $x = 2$	-1	-1
	0	0
	1	-1
To the right of $x = 2$	3	5.4
	4	4



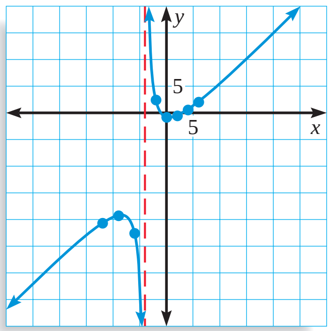
EXAMPLE 3 *Graphing a Rational Function ($m > n$)*

Graph $y = \dfrac{x^2 - 2x - 3}{x + 4}$.

SOLUTION

The numerator can be factored as $(x - 3)(x + 1)$, so the x-intercepts of the graph are 3 and -1. The only zero of the denominator is -4, so the only vertical asymptote is $x = -4$. The degree of the numerator (2) is greater than the degree of the denominator (1), so there is no horizontal asymptote and the end behavior of the graph of f is the same as the end behavior of the graph of $y = x^{2-1} = x$. To draw the graph, plot points to the left and right of the vertical asymptote.

	x	y
To the left of $x = -4$	-12	-20.6
	-9	-19.2
	-6	-22.5
To the right of $x = -4$	-2	2.5
	0	-0.75
	2	-0.5
	4	0.63
	6	2.1



GOAL 2 USING RATIONAL FUNCTIONS IN REAL LIFE

Manufacturers often want to package their products in a way that uses the least amount of packaging material. Finding the most efficient packaging sometimes involves finding a local minimum of a rational function.

REAL LIFE
Manufacturing

EXAMPLE 4 *Finding a Local Minimum*

A standard beverage can has a volume of 355 cubic centimeters.

a. Find the dimensions of the can that has this volume and uses the least amount of material possible.

b. Compare your result with the dimensions of an actual beverage can, which has a radius of 3.1 centimeters and a height of 11.8 centimeters.

SOLUTION

STUDENT HELP

Look Back
For help with rewriting an equation with more than one variable, see p. 26.

a. The volume must be 355 cubic centimeters, so you can write the height h of each possible can in terms of its radius r.

$V = \pi r^2 h$ **Formula for volume of cylinder**

$355 = \pi r^2 h$ **Substitute 355 for *V*.**

$\frac{355}{\pi r^2} = h$ **Solve for *h*.**

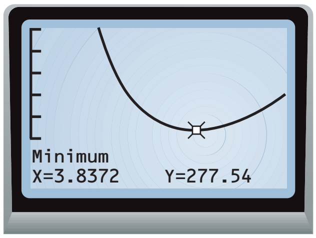
Using the least amount of material is equivalent to having a minimum surface area S. You can find the minimum surface area by writing its formula in terms of a single variable and graphing the result.

$S = 2\pi r^2 + 2\pi r h$ **Formula for surface area of cylinder**

$= 2\pi r^2 + 2\pi r\left(\frac{355}{\pi r^2}\right)$ **Substitute for *h*.**

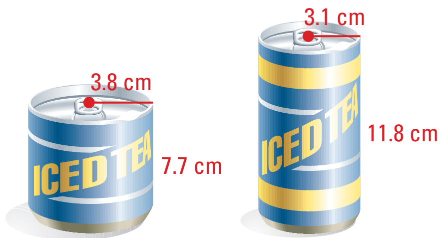
$= 2\pi r^2 + \frac{710}{r}$ **Simplify.**

Graph the function for the surface area S using a graphing calculator. Then use the *Minimum* feature to find the minimum value of S. When you do this, you get a minimum value of about 278, which occurs when $r \approx 3.84$ centimeters and



$$h \approx \frac{355}{\pi(3.84)^2} \approx 7.66 \text{ centimeters.}$$

b. An actual beverage can is taller and narrower than the can with minimal surface area—probably to make it easier to hold the can in one hand.



Can with minimal surface area **Actual beverage can**

Guided Practice

Vocabulary Check ✓

1. Let $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials with no common factors other than 1. Complete this statement: The line $y = 0$ is a horizontal asymptote of the graph of f when the degree of $q(x)$ is _?_ the degree of $p(x)$.

Concept Check ✓

2. Let $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials with no common factors other than 1. Describe how to find the x-intercepts and the vertical asymptotes of the graph of f.

3. Let $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are both cubic polynomials with no common factors other than 1. The leading coefficient of $p(x)$ is 8 and the leading coefficient of $q(x)$ is 2. Describe the end behavior of the graph of f.

Skill Check ✓

Graph the function.

4. $y = \frac{6}{x^2 + 3}$ **5.** $y = \frac{x^2 - 4}{x + 1}$ **6.** $y = \frac{x^2 - 7}{x^2 + 2}$

7. $y = \frac{x^3}{x^2 + 7}$ **8.** $y = \frac{2x^2}{x^2 - 1}$ **9.** $y = \frac{x}{x^2 - 16}$

10. **Soup Cans** The can for a popular brand of soup has a volume of about 342 cubic centimeters. Find the dimensions of the can with this volume that uses the least metal possible. Compare these dimensions with the dimensions of the actual can, which has a radius of 3.3 centimeters and a height of 10 centimeters.

Practice and Applications

Student Help

Extra Practice to help you master skills is on p. 952.

Analyzing Graphs **Identify the *x*-intercepts and vertical asymptotes of the graph of the function.**

11. $y = \frac{x}{x^2 - 9}$ **12.** $y = \frac{2x^2 + 3}{x - 1}$ **13.** $y = \frac{2x^2 - 9x - 5}{x^2 - 16}$

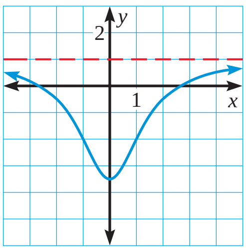
14. $y = \frac{2x + 3}{x^3}$ **15.** $y = \frac{x^2 + 4x - 5}{x - 6}$ **16.** $y = \frac{3x^2 - 13x + 10}{x^2 + 8}$

17. $y = \frac{2x + 8}{3x^2 - 9}$ **18.** $y = \frac{2x^2 + x}{x^2 + 1}$ **19.** $y = \frac{x^3 - 27}{2x}$

Matching Graphs **Match the function with its graph.**

20. $y = \frac{x^2 - 7}{x^2 + 2}$ **21.** $y = \frac{-8}{x^2 - 4}$ **22.** $y = \frac{x^3}{x^2 - 4}$

A.



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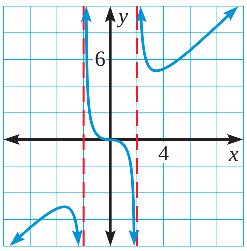
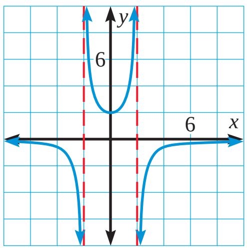


C.

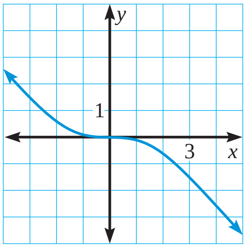


Student Help

Homework Help
Examples 1–3: Exs. 11–37
Example 4: Exs. 38–45

MATCHING GRAPHS **Match the function with its graph.**

23. $y = \dfrac{3}{x^3 - 27}$ **24.** $y = \dfrac{-x^3}{x^2 + 9}$ **25.** $y = \dfrac{x^2 + 4x}{2x - 1}$

A.


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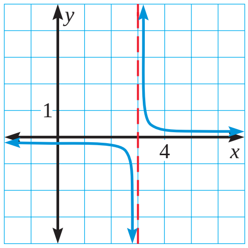

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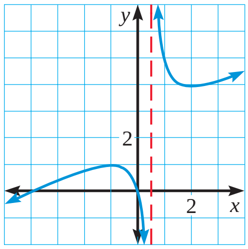


STUDENT HELP
INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 26–37.

GRAPHING FUNCTIONS **Graph the function.**

26. $y = \dfrac{2x^2 - 3}{x + 2}$ **27.** $y = \dfrac{-24}{x^2 + 8}$ **28.** $y = \dfrac{x^2 - 4}{x^2 + 3}$

29. $y = \dfrac{4x + 1}{x^2 - 1}$ **30.** $y = \dfrac{2x^2 + 3x + 1}{x^2 - 5x + 4}$ **31.** $y = \dfrac{-2x^2}{3x + 6}$

32. $y = \dfrac{3x^3 + 1}{4x^3 - 32}$ **33.** $y = \dfrac{x^2 - 11x - 12}{x^3 + 27}$ **34.** $y = \dfrac{4 - x}{5x^2 - 4x - 1}$

35. $y = \dfrac{-4x^2}{x^2 - 16}$ **36.** $y = \dfrac{x^2 - 9x + 20}{2x}$ **37.** $y = \dfrac{x^3 + 5x^2 - 1}{x^2 - 4x}$

38. **GARDEN FENCING** Suppose you want to make a rectangular garden with an area of 200 square feet. You want to use the side of your house for one side of the garden and use fencing for the other three sides. Find the dimensions of the garden that minimize the length of fencing needed.

GRAPHING MODELS **In Exercises 39–45, you may find it helpful to use a graphing calculator to graph the models.**

39. **ENERGY EXPENDITURE** The total energy expenditure E (in joules per gram mass per kilometer) of a typical budgerigar parakeet can be modeled by

$$E = \frac{0.31v^2 - 21.7v + 471.75}{v}$$

where v is the speed of the bird (in kilometers per hour). Graph the model. What speed minimizes a budgerigar's energy expenditure?

▶ Source: *Introduction to Mathematics for Life Scientists*

FOCUS ON CAREERS

REAL LIFE **HOSPITAL ADMINISTRATOR**
A hospital administrator oversees the quality of care and finances at a hospital. In 1996 there were approximately 329,000 hospital administrators in the United States.

INTERNET **CAREER LINK**
www.mcdougallittell.com

40. **OCEANOGRAPHY** The mean temperature T (in degrees Celsius) of the Atlantic Ocean between latitudes 40°N and 40°S can be modeled by

$$T = \frac{17{,}800d + 20{,}000}{3d^2 + 740d + 1000}$$

where d is the depth (in meters). Graph the model. Use your graph to estimate the depth at which the mean temperature is 4°C.

▶ Source: *Practical Handbook of Marine Science*

41. **HOSPITAL COSTS** For 1985 to 1995, the average daily cost per patient C (in dollars) at community hospitals in the United States can be modeled by

$$C = \frac{-22{,}407x + 462{,}048}{5x^2 - 122x + 1000}$$

where x is the number of years since 1985. Graph the model. Would you use this model to predict patient costs in 2005? Explain. ▶ Source: *Hospital Statistics*

42. **AUTOMOTIVE INDUSTRY** For 1980 to 1995, the total revenue R (in billions of dollars) from parking and automotive service and repair in the United States can be modeled by

$$R = \frac{427x^2 - 6416x + 30{,}432}{-0.7x^3 + 25x^2 - 268x + 1000}$$

where x is the number of years since 1980. Graph the model. In what year was the total revenue approximately \$75 billion?

INTERNET **DATA UPDATE** of U.S. Bureau of the Census data at www.mcdougallittell.com

SCIENCE CONNECTION **In Exercises 43–45, use the following information.**
The acceleration due to gravity g' (in meters per second squared) of a falling object at the moment it is dropped is given by the function

$$g' = \frac{3.99 \times 10^{14}}{h^2 + (1.28 \times 10^7)h + 4.07 \times 10^{13}}$$

where h is the object's altitude (in meters) above sea level.

43. Graph the function.

44. What is the acceleration due to gravity for an object dropped at an altitude of 5000 kilometers?

45. Describe what happens to g' as h increases.

46. **CRITICAL THINKING** Give an example of a rational function whose graph has two vertical asymptotes: $x = 2$ and $x = 7$.

Test Preparation

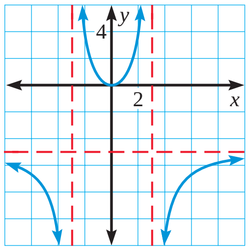
47. **MULTIPLE CHOICE** What is the horizontal asymptote of the graph of the following function?

$$y = \frac{10x^2 - 1}{x^3 + 8}$$

(A) $y = -10$ **(B)** $y = 0$ **(C)** $y = 2$

(D) $y = 10$ **(E)** No horizontal asymptote

48. **MULTIPLE CHOICE** Which of the following functions is graphed?



(A) $y = \dfrac{-5x^2}{x^2 + 9}$ **(B)** $y = \dfrac{5x^2}{x^2 - 9}$

(C) $y = \dfrac{5x^2}{x^2 + 9}$ **(D)** $y = \dfrac{-5x^2}{x^2 - 9}$

★ Challenge

49. Consider the following two functions:

$$f(x) = \frac{(x + 1)(x + 2)}{(x - 3)(x - 5)} \quad \text{and} \quad g(x) = \frac{(x + 2)(x - 3)}{(x - 3)(x - 5)}$$

Notice that the numerator and denominator of g have a common factor of $x - 3$.

a. Make a table of values for each function from $x = 2.95$ to $x = 3.05$ in increments of 0.01.

b. Use your table of values to graph each function for $2.95 \le x \le 3.05$.

c. As x approaches 3, what happens to the graph of $f(x)$? to the graph of $g(x)$?

d. What do you think is true about the graph of a function $g(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ have a common factor $x - k$?

EXTRA CHALLENGE
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MIXED REVIEW

SIMPLIFYING ALGEBRAIC EXPRESSIONS **Simplify the expression. Tell which properties of exponents you used. (Review 6.1 for 9.4)**

50. $\dfrac{x^{-3}y}{xy^4}$ **51.** $\dfrac{x^6y^5}{xy}$ **52.** $\dfrac{3x^3y^3}{6x^{-1}y}$

53. $\dfrac{12x^5y^{-2}}{3x^{-2}y^5}$ **54.** $\left(\dfrac{x^2y^2}{x^3y}\right)^2$ **55.** $\left(\dfrac{5x^3}{25xy^2}\right)^3$

JOINT VARIATION MODELS **The variable z varies jointly with x and y. Use the given values to write an equation relating x, y, and z. Then find z when $x = -3$ and $y = 2$. (Review 9.1)**

56. $x = 3,\ y = -6,\ z = 2$ **57.** $x = -5,\ y = 2,\ z = \frac{3}{4}$

58. $x = -8,\ y = 4,\ z = \frac{8}{3}$ **59.** $x = 1,\ y = \frac{1}{2},\ z = 4$

VERIFYING INVERSES **Verify that f and g are inverse functions. (Review 7.4)**

60. $f(x) = \frac{1}{2}x - 3,\ g(x) = 2x + 6$ **61.** $f(x) = -3x + 2,\ g(x) = -\frac{1}{3}x + \frac{2}{3}$

62. $f(x) = 5x^3 + 2,\ g(x) = \left(\dfrac{x-2}{5}\right)^{1/3}$ **63.** $f(x) = 16x^4,\ x \ge 0;\ g(x) = \dfrac{\sqrt[4]{x}}{2}$

QUIZ 1

Self-Test for Lessons 9.1–9.3

The variables x and y vary inversely. Use the given values to write an equation relating x and y. Then find y when $x = -3$. (Lesson 9.1)

1. $x = 6,\ y = -2$ **2.** $x = 11,\ y = 6$ **3.** $x = \frac{1}{5},\ y = 30$

The variable x varies jointly with y and z. Use the given values to write an equation relating x, y, and z. Then find y when $x = 4$ and $z = 1$. (Lesson 9.1)

4. $x = 5,\ y = -5,\ z = 6$ **5.** $x = 12,\ y = 6,\ z = \frac{1}{2}$ **6.** $x = -10,\ y = 2,\ z = 4$

Graph the function. (Lessons 9.2 and 9.3)

7. $y = \dfrac{10}{x}$ **8.** $y = \dfrac{2}{x+9} - 7$ **9.** $y = \dfrac{3x+5}{2x-11}$

10. $y = \dfrac{6x}{x^2 - 36}$ **11.** $y = \dfrac{3x^2}{x^2 - 25}$ **12.** $y = \dfrac{x^2 - 4x - 5}{x+2}$

13. **HOTEL REVENUE** For 1980 to 1995, the total revenue R (in billions of dollars) from hotels and motels in the United States can be modeled by

$$R = \frac{2.76x + 26.88}{-0.01x + 1}$$

where x is the number of years since 1980. Graph the model. Use your graph to find the year in which the total revenue from hotels and motels was approximately \$68 billion. **(Lesson 9.2)**