

9.1

Inverse and Joint Variation

GOAL 1 USING INVERSE VARIATION

In Lesson 2.4 you learned that two variables x and y show direct variation if $y = kx$ for some nonzero constant k . Another type of variation is called *inverse variation*. Two variables x and y show **inverse variation** if they are related as follows:

$$y = \frac{k}{x}, k \neq 0$$

The nonzero constant k is called the **constant of variation**, and y is said to *vary inversely* with x .

EXAMPLE 1 Classifying Direct and Inverse Variation

Tell whether x and y show *direct variation*, *inverse variation*, or *neither*.

GIVEN EQUATION	REWRITTEN EQUATION	TYPE OF VARIATION
a. $\frac{y}{5} = x$	$y = 5x$	Direct
b. $y = x + 2$		Neither
c. $xy = 4$	$y = \frac{4}{x}$	Inverse

EXAMPLE 2 Writing an Inverse Variation Equation

The variables x and y vary inversely, and $y = 8$ when $x = 3$.

- Write an equation that relates x and y .
- Find y when $x = -4$.

SOLUTION

- Use the given values of x and y to find the constant of variation.

$$y = \frac{k}{x} \quad \text{Write general equation for inverse variation.}$$

$$8 = \frac{k}{3} \quad \text{Substitute 8 for } y \text{ and 3 for } x.$$

$$24 = k \quad \text{Solve for } k.$$

- The inverse variation equation is $y = \frac{24}{x}$.

- When $x = -4$, the value of y is:

$$\begin{aligned} y &= \frac{24}{-4} \\ &= -6 \end{aligned}$$

What you should learn

GOAL 1 Write and use inverse variation models, as applied in **Example 4**.

GOAL 2 Write and use joint variation models, as applied in **Example 6**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the speed of a whirlpool's current in **Example 3**.





EXAMPLE 3 Writing an Inverse Variation Model

The speed of the current in a whirlpool varies inversely with the distance from the whirlpool's center. The Lofoten Maelstrom is a whirlpool located off the coast of Norway. At a distance of 3 kilometers (3000 meters) from the center, the speed of the current is about 0.1 meter per second. Describe the change in the speed of the current as you move closer to the whirlpool's center.

SOLUTION

First write an inverse variation model relating distance from center d and speed s .

$$s = \frac{k}{d} \quad \text{Model for inverse variation}$$

$$0.1 = \frac{k}{3000} \quad \text{Substitute 0.1 for } s \text{ and 3000 for } d.$$

$$300 = k \quad \text{Solve for } k.$$

The model is $s = \frac{300}{d}$. The table shows some speeds for different values of d .

Distance from center (meters), d	2000	1500	500	250	50
Speed (meters per second), s	0.15	0.2	0.6	1.2	6

► From the table you can see that the speed of the current increases as you move closer to the whirlpool's center.

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The equation for inverse variation can be rewritten as $xy = k$. This tells you that a set of data pairs (x, y) shows inverse variation if the products xy are constant or approximately constant.

EXAMPLE 4 Checking Data for Inverse Variation

BIOLOGY CONNECTION The table compares the wing flapping rate r (in beats per second) to the wing length l (in centimeters) for several birds. Do these data show inverse variation? If so, find a model for the relationship between r and l .

Bird	r (beats per second)	l (cm)
Carrion crow	3.6	32.5
Common scoter	5.0	23.5
Great crested grebe	6.3	18.7
Curlew	4.0	29.2
Lesser black-backed gull	2.8	42.2

► Source: *Smithsonian Miscellaneous Collections*

SOLUTION

Each product rl is approximately equal to 117. For instance, $(3.6)(32.5) = 117$ and $(5.0)(23.5) = 117.5$. So, the data do show inverse variation. A model for the relationship between wing flapping rate and wing length is $r = \frac{117}{l}$.

FOCUS ON APPLICATIONS



COMMON SCOTER The common scoter migrates from the Quebec/Labrador border in Canada to coastal cities such as Portland, Maine, and Galveston, Texas. To reach its winter destination, the scoter will travel up to 2150 miles.

GOAL 2 USING JOINT VARIATION

STUDENT HELP

Look Back

For help with direct variation, see p. 94.

Joint variation occurs when a quantity varies directly as the product of *two or more* other quantities. For instance, if $z = kxy$ where $k \neq 0$, then z varies jointly with x and y . Other types of variation are also possible, as illustrated in the following example.

EXAMPLE 5 Comparing Different Types of Variation

Write an equation for the given relationship.

RELATIONSHIP	EQUATION
a. y varies directly with x .	$y = kx$
b. y varies inversely with x .	$y = \frac{k}{x}$
c. z varies jointly with x and y .	$z = kxy$
d. y varies inversely with the square of x .	$y = \frac{k}{x^2}$
e. z varies directly with y and inversely with x .	$z = \frac{ky}{x}$

EXAMPLE 6 Writing a Variation Model

SCIENCE CONNECTION The *law of universal gravitation* states that the gravitational force F (in newtons) between two objects varies jointly with their masses m_1 and m_2 (in kilograms) and inversely with the square of the distance d (in meters) between the two objects. The constant of variation is denoted by G and is called the *universal gravitational constant*.

- Write an equation for the law of universal gravitation.
- Estimate the universal gravitational constant. Use the Earth and sun facts given at the right.

Mass of Earth:
 $m_1 = 5.98 \times 10^{24}$ kg

Mass of sun:
 $m_2 = 1.99 \times 10^{30}$ kg

Mean distance between Earth and sun:
 $d = 1.50 \times 10^{11}$ m

Force between Earth and sun:
 $F = 3.53 \times 10^{22}$ N

SOLUTION

a. $F = \frac{Gm_1m_2}{d^2}$

- b. Substitute the given values and solve for G .

$$F = \frac{Gm_1m_2}{d^2}$$

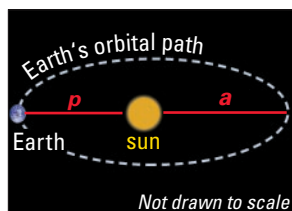
$$3.53 \times 10^{22} = \frac{G(5.98 \times 10^{24})(1.99 \times 10^{30})}{(1.50 \times 10^{11})^2}$$

$$3.53 \times 10^{22} \approx G(5.29 \times 10^{32})$$

$$6.67 \times 10^{-11} \approx G$$

- ▶ The universal gravitational constant is about $6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}$.

FOCUS ON APPLICATIONS



Not drawn to scale

REAL LIFE EARTH AND SUN

Earth's orbit around the sun is elliptical, so its distance from the sun varies. The shortest distance p is 1.47×10^{11} meters and the longest distance a is 1.52×10^{11} meters.

APPLICATION LINK

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GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: If w varies directly as the product of x , y , and z , then w varies ? with x , y , and z .

Concept Check ✓

2. How can you tell whether a set of data pairs (x, y) shows inverse variation?

3. Suppose z varies jointly with x and y . What can you say about $\frac{z}{xy}$?

Skill Check ✓

Tell whether x and y show **direct variation**, **inverse variation**, or **neither**.

4. $xy = \frac{1}{4}$

5. $\frac{x}{y} = 5$

6. $y = x - 3$

7. $x = \frac{7}{y}$

8. $\frac{y}{x} = 12$

9. $\frac{1}{2}xy = 9$

10. $y = \frac{1}{x}$

11. $2x + y = 4$

Tell whether x varies jointly with y and z .

12. $x = 15yz$

13. $\frac{x}{z} = 0.5y$

14. $xy = 4z$


15. $x = \frac{yz}{2}$

16. $x = \frac{3z}{y}$

17. $2yz = 7x$

18. $\frac{x}{y} = 17z$

19. $5x = 4yz$

20.  **TOOLS** The force F needed to loosen a bolt with a wrench varies inversely with the length l of the handle. Write an equation relating F and l , given that 250 pounds of force must be exerted to loosen a bolt when using a wrench with a handle 6 inches long. How much force must be exerted when using a wrench with a handle 24 inches long?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice
to help you master skills is on p. 952.

DETERMINING VARIATION Tell whether x and y show **direct variation**, **inverse variation**, or **neither**.

21. $xy = 10$

22. $xy = \frac{1}{10}$

23. $y = x - 1$

24. $\frac{y}{9} = x$

25. $x = \frac{5}{y}$

26. $3x = y$

27. $x = 5y$

28. $x + y = 2.5$

INVERSE VARIATION MODELS The variables x and y vary inversely. Use the given values to write an equation relating x and y . Then find y when $x = 2$.

29. $x = 5, y = -2$

30. $x = 4, y = 8$

31. $x = 7, y = 1$

32. $x = \frac{1}{2}, y = 10$

33. $x = -\frac{2}{3}, y = 6$

34. $x = \frac{3}{4}, y = \frac{3}{8}$

INTERPRETING DATA Determine whether x and y show **direct variation**, **inverse variation**, or **neither**.

35.

x	y
1.5	20
2.5	12
4	7.5
5	6

36.

x	y
31	217
20	140
17	119
12	84

37.

x	y
3	36
7	105
5	50
16	48

38.

x	y
4	16
5	12.8
1.6	40
20	3.2

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 21–28

Example 2: Exs. 29–34

Example 3: Exs. 51–54

Example 4: Exs. 35–38,
48, 49

Example 5: Exs. 45–47

Example 6: Exs. 55–58

JOINT VARIATION MODELS The variable z varies jointly with x and y . Use the given values to write an equation relating x , y , and z . Then find z when $x = -4$ and $y = 7$.

39. $x = 3, y = 8, z = 6$

40. $x = -12, y = 4, z = 2$

41. $x = 1, y = \frac{1}{3}, z = 5$

42. $x = -6, y = 3, z = \frac{2}{5}$

43. $x = \frac{5}{6}, y = \frac{3}{10}, z = 8$

44. $x = \frac{3}{8}, y = \frac{16}{17}, z = \frac{3}{2}$

STUDENT HELP



HOMEWORK HELP

Visit our Web site
www.mcdougallittell.com
for help with Exs. 45–47.

WRITING EQUATIONS Write an equation for the given relationship.

45. x varies inversely with y and directly with z .

46. y varies jointly with z and the square root of x .

47. w varies inversely with x and jointly with y and z .

HOME REPAIR In Exercises 48–50, use the following information.

On some tubes of caulking, the diameter of the circular nozzle opening can be adjusted to produce lines of varying thickness. The table shows the length l of caulking obtained from a tube when the nozzle opening has diameter d and cross-sectional area A .

d (in.)	A (in. ²)	l (in.)
$\frac{1}{8}$	$\frac{\pi}{256}$	1440
$\frac{1}{4}$	$\frac{\pi}{64}$	360
$\frac{3}{8}$	$\frac{9\pi}{256}$	160
$\frac{1}{2}$	$\frac{\pi}{16}$	90

48. Determine whether l varies inversely with d . If so, write an equation relating l and d .

49. Determine whether l varies inversely with A . If so, write an equation relating l and A .

50. Find the length of caulking you get from a tube whose nozzle opening has a diameter of $\frac{3}{4}$ inch.

ASTRONOMY In Exercises 51–53, use the following information.

A star's diameter D (as a multiple of the sun's diameter) varies directly with the square root of the star's luminosity L (as a multiple of the sun's luminosity) and inversely with the square of the star's temperature T (in kelvins).

51. Write an equation relating D , L , T , and a constant k .

52. The luminosity of Polaris is 10,000 times the luminosity of the sun. The surface temperature of Polaris is about 5800 kelvins. Using $k = 33,640,000$, find how the diameter of Polaris compares with the diameter of the sun.

53. The sun's diameter is 1,390,000 kilometers. What is the diameter of Polaris?

54. **INTENSITY OF SOUND** The intensity I of a sound (in watts per square meter) varies inversely with the square of the distance d (in meters) from the sound's source. At a distance of 1 meter from the stage, the intensity of the sound at a rock concert is about 10 watts per square meter. Write an equation relating I and d . If you are sitting 15 meters back from the stage, what is the intensity of the sound you hear?

55. **SCIENCE CONNECTION** The work W (in joules) done when lifting an object varies jointly with the mass m (in kilograms) of the object and the height h (in meters) that the object is lifted. The work done when a 120 kilogram object is lifted 1.8 meters is 2116.8 joules. Write an equation that relates W , m , and h . How much work is done when lifting a 100 kilogram object 1.5 meters?

FOCUS ON PEOPLE



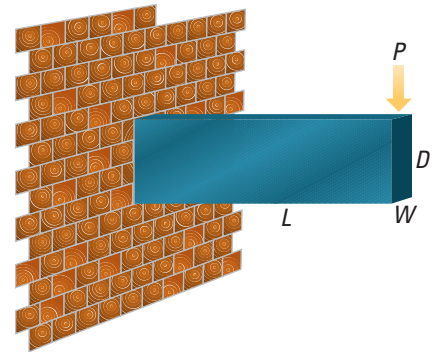
STEPHEN HAWKING, a theoretical physicist, has spent years studying *black holes*. A black hole is believed to be formed when a star's core collapses. The gravitational pull becomes so strong that even the star's light, as discussed in Exs. 51–53, cannot escape.

 **HEAT LOSS** In Exercises 56 and 57, use the following information.

The heat loss h (in watts) through a single-pane glass window varies jointly with the window's area A (in square meters) and the difference between the inside and outside temperatures d (in kelvins).

56. Write an equation relating h , A , d , and a constant k .
57. A single-pane window with an area of 1 square meter and a temperature difference of 1 kelvin has a heat loss of 5.7 watts. What is the heat loss through a single-pane window with an area of 2.5 square meters and a temperature difference of 20 kelvins?
58. **GEOMETRY CONNECTION** The area of a trapezoid varies jointly with the height and the sum of the lengths of the bases. When the sum of the lengths of the bases is 18 inches and the height is 4 inches, the area is 36 square inches. Find a formula for the area of a trapezoid.
59. **MULTI-STEP PROBLEM** The load P (in pounds) that can be safely supported by a horizontal beam varies jointly with the width W (in feet) of the beam and the square of its depth D (in feet), and inversely with its length L (in feet).

- a. How does P change when the width and length of the beam are doubled?
- b. How does P change when the width and depth of the beam are doubled?
- c. How does P change when all three dimensions are doubled?
- d. *Writing* Describe several ways a beam can be modified if the safe load it is required to support is increased by a factor of 4.



Test Preparation 

★ Challenge

60. **LOGICAL REASONING** Suppose x varies inversely with y and y varies inversely with z . How does x vary with z ? Justify your answer algebraically.

MIXED REVIEW

SQUARE ROOT FUNCTIONS Graph the function. Then state the domain and range. (Review 7.5 for 9.2)

61. $y = \sqrt{x + 2}$

62. $y = \sqrt{x} - 4$

63. $y = \sqrt{x + 1} - 3$

SOLVING RADICAL EQUATIONS Solve the equation. Check for extraneous solutions. (Review 7.6)

64. $\sqrt{x} = 22$


65. $\sqrt[4]{2x} + 2 = 6$

66. $x^{1/3} - 7 = 0$

67. $\sqrt[3]{x + 12} = 5$

68. $(x - 2)^{3/2} = -8$

69. $\sqrt{3x + 1} = \sqrt{x + 15}$

70.  **COLLEGE ADMISSION** The number of admission applications received by a college was 1152 in 1990 and increased 5% per year until 1998. (Review 8.1 for 9.2)

- a. Write a model giving the number A of applications t years after 1990.
- b. Graph the model. Use the graph to estimate the year in which there were 1400 applications.