8.8 Logistic Growth Functions

What *you should learn*

GOAL 1 Evaluate and graph logistic growth functions.

GOAL 2 Use logistic growth functions to model **real-life** quantities, such as a yeast population in **Exs. 50 and 51**.

Why *you should learn it*

▼ To solve **real-life** problems, such as modeling the height of a sunflower in **Example 5**.

REAL LIFE

GOAL 1 USING LOGISTIC GROWTH FUNCTIONS

In this lesson you will study a family of functions of the form

$$y = \frac{c}{1 + ae^{-rx}}$$

where a, c, and r are all positive constants. Functions of this form are called **logistic growth functions**.

EXAMPLE 1 *Evaluating a Logistic Growth Function*

Evaluate $f(x) = \dfrac{100}{1 + 9e^{-2x}}$ for each value of x.

a. $f(-3)$ **b.** $f(0)$ **c.** $f(2)$ **d.** $f(4)$

SOLUTION

a. $f(-3) = \dfrac{100}{1 + 9e^{-2(-3)}} \approx 0.0275$

b. $f(0) = \dfrac{100}{1 + 9e^{-2(0)}} = \dfrac{100}{10} = 10$

c. $f(2) = \dfrac{100}{1 + 9e^{-2(2)}} \approx 85.8$

d. $f(4) = \dfrac{100}{1 + 9e^{-2(4)}} \approx 99.7$

ACTIVITY Developing Concepts

Graphs of Logistic Growth Functions

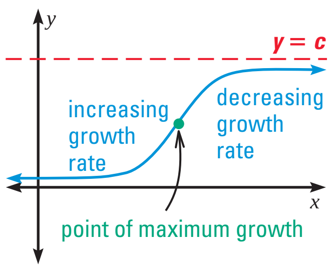
1. Use a graphing calculator to graph the logistic growth function from Example 1. Trace along the graph to determine the function's end behavior.

2. Use a graphing calculator to graph each of the following. Then describe the basic shape of the graph of a logistic growth function.

a. $y = \dfrac{1}{1 + e^{-x}}$ **b.** $y = \dfrac{10}{1 + 5e^{-2x}}$ **c.** $y = \dfrac{5}{1 + 10e^{-2x}}$

In this chapter you learned that an exponential growth function $f(x)$ increases without bound as x increases. On the other hand, the logistic growth function $y = \dfrac{c}{1 + ae^{-rx}}$ has $y = c$ as an upper bound.

Logistic growth functions are used to model real-life quantities whose growth levels off because the rate of growth changes—from an increasing growth rate to a decreasing growth rate.



GRAPHS OF LOGISTIC GROWTH FUNCTIONS

The graph of $y = \dfrac{c}{1 + ae^{-rx}}$ has the following characteristics:

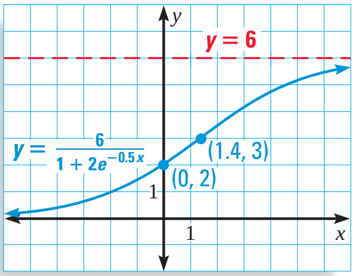
- The horizontal lines $y = 0$ and $y = c$ are asymptotes.
- The y-intercept is $\dfrac{c}{1 + a}$.
- The domain is all real numbers, and the range is $0 < y < c$.
- The graph is increasing from left to right. To the left of its point of maximum growth, $\left(\dfrac{\ln a}{r}, \dfrac{c}{2}\right)$, the rate of increase is increasing. To the right of its point of maximum growth, the rate of increase is decreasing.

EXAMPLE 2 *Graphing a Logistic Growth Function*

Graph $y = \dfrac{6}{1 + 2e^{-0.5x}}$.

SOLUTION

Begin by sketching the upper horizontal asymptote, $y = 6$. Then plot the y-intercept at $(0, 2)$ and the point of maximum growth $\left(\dfrac{\ln 2}{0.5}, \dfrac{6}{2}\right) \approx (1.4, 3)$. Finally, from left to right, draw a curve that starts just above the x-axis, curves up to the point of maximum growth, and then levels off as it approaches the upper horizontal asymptote.



EXAMPLE 3 *Solving a Logistic Growth Equation*

Solve $\dfrac{50}{1 + 10e^{-3x}} = 40$.

SOLUTION

$\dfrac{50}{1 + 10e^{-3x}} = 40$	Write original equation.
$50 = (1 + 10e^{-3x})(40)$	Multiply each side by $1 + 10e^{-3x}$.
$50 = 40 + 400e^{-3x}$	Use distributive property.
$10 = 400e^{-3x}$	Subtract 40 from each side.
$0.025 = e^{-3x}$	Divide each side by 400.
$\ln 0.025 = -3x$	Take natural log of each side.
$-\dfrac{1}{3}\ln 0.025 = x$	Divide each side by -3.
$1.23 \approx x$	Use a calculator.

▶ The solution is about 1.23. Check this in the original equation.

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING LOGISTIC GROWTH MODELS IN REAL LIFE

Logistic growth functions are often more useful as models than exponential growth functions because they account for constraints placed on the growth. An example is a bacteria culture allowed to grow under initially ideal conditions, followed by less favorable conditions that inhibit growth.

REAL LIFE Biology

EXAMPLE 4 *Using a Logistic Growth Model*

A colony of the bacteria *B. dendroides* is growing in a petri dish. The colony's area A (in square centimeters) can be modeled by

$$A = \frac{49.9}{1 + 134e^{-1.96t}}$$

where t is the elapsed time in days. Graph the function and describe what it tells you about the growth of the bacteria colony.

SOLUTION

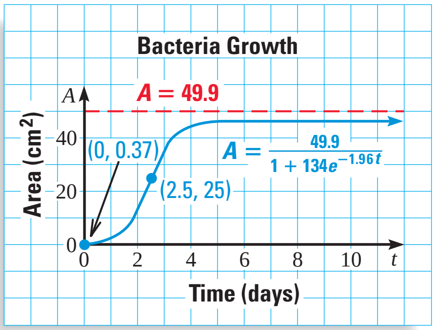
The graph of the model is shown. The initial area is

$$A = \frac{49.9}{1 + 134e^{-1.96(0)}} \approx 0.37 \text{ cm}^2.$$

The colony grows more and more rapidly until

$$t = \frac{\ln 134}{1.96} \approx 2.5 \text{ days.}$$

Then the rate of growth decreases. The colony's area is limited to $A = 49.9 \text{ cm}^2$, which might possibly be the area of the petri dish.



REAL LIFE Botany

EXAMPLE 5 *Writing a Logistic Growth Model*

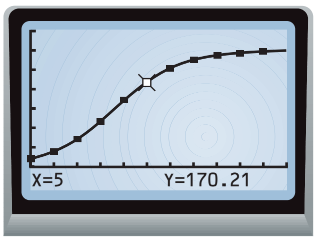
You planted a sunflower seedling and kept track of its height h (in centimeters) over time t (in weeks). Find a model that gives h as a function of t.

t	0	1	2	3	4	5	6	7	8	9	10
h	18	33	56	90	130	170	203	225	239	247	251

SOLUTION

A scatter plot shows that the data can be modeled by a logistic growth function.

The logistic regression feature of a graphing calculator returns the values shown at the right.



▶ The model is:

$$h = \frac{256}{1 + 13e^{-0.65t}}$$

STUDENT HELP

INTERNET KEYSTROKE HELP
Visit our Web site www.mcdougallittell.com to see keystrokes for several models of calculators.

Guided Practice

Vocabulary Check ✓

1. What is the name of a function having the form $y = \frac{c}{1 + ae^{-rx}}$ where c, a, and r are positive constants?

Concept Check ✓

2. What is a significant difference between using exponential growth functions and using logistic growth functions as models for real-life quantities?

3. What is the significance of the point $(\ln 3, 4)$ on the graph of $f(x) = \frac{8}{1 + 3e^{-x}}$?

Skill Check ✓

Evaluate the function $f(x) = \frac{12}{1 + 5e^{-2x}}$ for the given value of x.

4. $f(0)$ 5. $f(-2)$ 6. $f(5)$ 7. $f\left(-\frac{1}{2}\right)$ 8. $f(10)$

Graph the function. Identify the asymptotes, y-intercept, and point of maximum growth.

9. $f(x) = \frac{5}{1 + 4e^{-2.5x}}$ 10. $f(x) = \frac{8}{1 + 3e^{-0.4x}}$ 11. $f(x) = \frac{2}{1 + 4e^{-0.25x}}$

Solve the equation.

12. $\frac{18}{1 + 2e^{-2x}} = 10$ 13. $\frac{30}{1 + 4e^{-x}} = 10$ 14. $\frac{12.5}{1 + 7e^{-0.2x}} = 9$

15. **Planting Seeds** You planted a seedling and kept track of its height h (in centimeters) over time t (in weeks). Use the data in the table to find a model that gives h as a function of t.

t	0	1	2	3	4	5	6	7	8
h	5	12	26	39	51	88	94	103	112

Practice and Applications

Student Help

Extra Practice to help you master skills is on p. 952.

Evaluating Functions Evaluate the function $f(x) = \frac{7}{1 + 3e^{-x}}$ for the given value of x.

16. $f(1)$ 17. $f(3)$ 18. $f(-1)$ 19. $f(-6)$

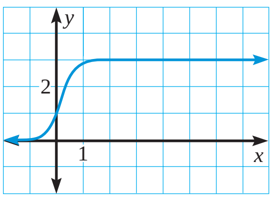
20. $f(0)$ 21. $f\left(\frac{3}{4}\right)$ 22. $f(2.2)$ 23. $f(-0.9)$

Matching Graphs Match the function with its graph.

24. $f(x) = \frac{4}{1 + 2e^{-3x}}$ 25. $f(x) = \frac{3}{1 + 2e^{-4x}}$ 26. $f(x) = \frac{2}{1 + 3e^{-4x}}$

Student Help

Homework Help
Example 1: Exs. 16–23
Example 2: Exs. 24–35
Example 3: Exs. 36–44
Example 4: Exs. 45–49
Example 5: Exs. 50, 51

A.


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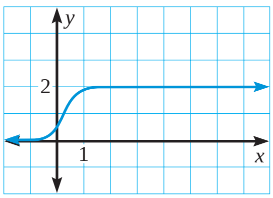

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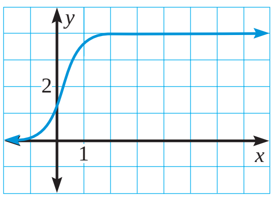

GRAPHING FUNCTIONS **Graph the function. Identify the asymptotes, *y*-intercept, and point of maximum growth.**

27. $y = \dfrac{1}{1 + 6e^{-x}}$ **28.** $y = \dfrac{2}{1 + 0.4e^{-0.3x}}$ **29.** $y = \dfrac{5}{1 + e^{-10x}}$

30. $y = \dfrac{4}{1 + 0.08e^{-2.1x}}$ **31.** $y = \dfrac{4}{1 + 3e^{-3x}}$ **32.** $y = \dfrac{3}{1 + 3e^{-8x}}$

33. $y = \dfrac{8}{1 + e^{-1.02x}}$ **34.** $y = \dfrac{10}{1 + 6e^{-0.5x}}$ **35.** $y = \dfrac{6}{1 + 0.8e^{-2x}}$

SOLVING EQUATIONS **Solve the equation.**

36. $\dfrac{8}{1 + 3e^{-x}} = 5$ **37.** $\dfrac{10}{1 + 2e^{-4x}} = 9$ **38.** $\dfrac{3}{1 + 18e^{-x}} = 1$

39. $\dfrac{28}{1 + 13e^{-2x}} = 20$ **40.** $\dfrac{82}{1 + 50e^{-x}} = 68$ **41.** $\dfrac{36}{1 + 7e^{-10x}} = 30$

42. $\dfrac{41}{1 + 14.9e^{-6x}} = 7$ **43.** $\dfrac{9}{1 + 5e^{-0.2x}} = \dfrac{3}{4}$ **44.** $\dfrac{40}{1 + 2.5e^{-0.4x}} = 6.4$

OWNING A VCR **In Exercises 45–47, use the following information.**
The number of households in the United States that own VCRs has shown logistic growth from 1980 through 1999. The number H (in millions) of households can be modeled by the equation

$$H = \frac{91.86}{1 + 22.96e^{-0.4t}}$$

where t is the number of years since 1980. ▶ Source: Veronis, Suhler & Associates

45. In what year were there approximately 86 million households with VCRs?

46. Graph the model. In what year did the growth rate for the number of households stop increasing and start decreasing?

47. What is the long-term trend in VCR ownership?

ECONOMICS **In Exercises 48 and 49, use the following information.**
The gross domestic product (GDP) of the United States has shown logistic growth from 1970 through 1992. The gross domestic product G (in billions of dollars) can be modeled by the equation

$$G = \frac{9200}{1 + 8.03e^{-0.121t}}$$

where t is the number of years since 1970. ▶ Source: U.S. Bureau of the Census

48. In what year was the GDP approximately \$5000 billion?

49. Graph the model. When did the GDP reach its point of maximum growth?

YEAST POPULATION **In Exercises 50 and 51, use the following information.**
In biology class, you observed the biomass of a yeast population over a period of time. The table gives the yeast mass y (in grams) after t hours.

t	0	1	2	3	4	5	6	7	8	9
y	9.6	18.3	29.0	47.2	71.1	119.1	174.6	257.3	350.7	441.0

50. Draw a scatter plot of the data.

51. Find a model that gives y as a function of t using the logistic regression feature of a graphing calculator.

FOCUS ON APPLICATIONS

REAL LIFE **ECONOMICS**
Gross domestic product, the focus of Exs. 48 and 49, is the value of all goods and services produced within a country during a given period.

Test Preparation

52. **MULTI-STEP PROBLEM** The table shows the population P (in millions) of the United States from 1800 to 1870 where t represents the number of years since 1800. ▶ Source: U.S. Bureau of the Census

t	P
0	5.3
10	7.2
20	9.6
30	12.9
40	17.0
50	23.2
60	31.4
70	39.8

a. Use a graphing calculator to find an exponential growth model and a logistic growth model for the data. Then graph both models.

b. Use the models from part (a) to find the year when the population was about 92 million. Which of the models gives a year that is closer to 1910, the correct answer? Explain why you think that model is more accurate.

c. Use each model to predict the population in 2010. Which model gives a population closer to 297.7 million, the predicted population from the U.S. Bureau of the Census?

STUDENT HELP

INTERNET

DATA UPDATE
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★ **Challenge**

53. **ANALYZING MODELS** The graph of a logistic growth function $y = \frac{c}{1 + ae^{-rx}}$ reaches its point of maximum growth where $y = \frac{c}{2}$. Show that the x-coordinate of this point is $x = \frac{\ln a}{r}$.

MIXED REVIEW

WRITING EQUATIONS The variables x and y vary directly. Write an equation that relates the variables. (Review 2.4 for 9.1)

54. $x = 4, y = 36$ **55.** $x = -5, y = 10$ **56.** $x = 2, y = 13$

57. $x = 40, y = 5$ **58.** $x = 0.1, y = 0.9$ **59.** $x = 1, y = 0.2$

WRITING EQUATIONS Write y as a function of x. (Review 8.7)

60. $\log y = 0.9 \log x + 2.11$ **61.** $\ln y = 0.94 - 2.44x$

62. $\log y = -1.82 + 0.4x$ **63.** $\log y = -0.75 \log x - 1.76$

QUIZ 3

Self-Test for Lessons 8.7 and 8.8

Write an exponential function of the form $y = ab^x$ whose graph passes through the given points. (Lesson 8.7)

1. (2, 3), (5, 12) **2.** (1, 16), (3, 45) **3.** (5, 9), (8, 35)

Write a power function of the form $y = ax^b$ whose graph passes through the given points. (Lesson 8.7)

4. (2, 28), (8, 192) **5.** (1, 0.5), (6, 48) **6.** (5, 40), (2, 6)

7. **FLU VIRUS** The spread of a virus through a student population can be modeled by $S = \frac{5000}{1 + 4999e^{-0.8t}}$ where S is the total number of students infected after t days. Graph the model and tell when the point of maximum growth in infections is reached. (Lesson 8.8)