EXPLORING DATA AND STATISTICS

8.7

Modeling with Exponential and Power Functions

GOAL 1 MODELING WITH EXPONENTIAL FUNCTIONS

What *you should learn*

GOAL 1 Model data with exponential functions.

GOAL 2 Model data with power functions, as applied in **Example 5**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding the number of U.S. stamps issued in **Ex. 56**.

REAL LIFE

Just as two points determine a line, two points also determine an exponential curve.

EXAMPLE 1 *Writing an Exponential Function*

Write an exponential function $y = ab^x$ whose graph passes through (1, 6) and (3, 24).

SOLUTION

Substitute the coordinates of the two given points into $y = ab^x$ to obtain two equations in a and b.

$6 = ab^1$ **Substitute 6 for *y* and 1 for *x*.**

$24 = ab^3$ **Substitute 24 for *y* and 3 for *x*.**

To solve the system, solve for a in the first equation to get $a = \frac{6}{b}$, then substitute into the second equation.

$24 = \left(\frac{6}{b}\right)b^3$ **Substitute $\frac{6}{b}$ for *a*.**

$24 = 6b^2$ **Simplify.**

$4 = b^2$ **Divide each side by 6.**

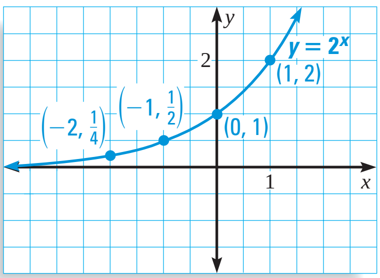
$2 = b$ **Take the positive square root.**

Using $b = 2$, you then have $a = \frac{6}{b} = \frac{6}{2} = 3$. So, $y = 3 \cdot 2^x$.

. . . .

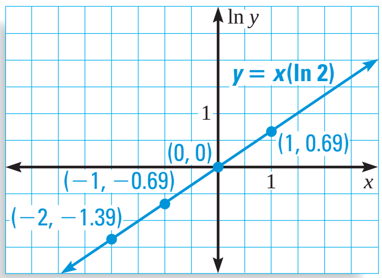
When you are given more than two points, you can decide whether an exponential model fits the points by plotting the natural logarithms of the y-values against the x-values. If the new points $(x, \ln y)$ fit a linear pattern, then the original points (x, y) fit an exponential pattern.

Graph of points (x, y)



The graph is an exponential curve.

Graph of points $(x, \ln y)$



The graph is a line.

REAL LIFE
Communications

EXAMPLE 2 *Finding an Exponential Model*

The table gives the number y (in millions) of cell-phone subscribers from 1988 to 1997 where t is the number of years since 1987.

t	1	2	3	4	5	6	7	8	9	10
y	1.6	2.7	4.4	6.4	8.9	13.1	19.3	28.2	38.2	48.7

▶ Source: Cellular Telecommunications Industry Association

STUDENT HELP

Look Back
For help with scatter plots and best-fitting lines, see pp. 100–101.

a. Draw a scatter plot of ln y versus x. Is an exponential model a good fit for the original data?

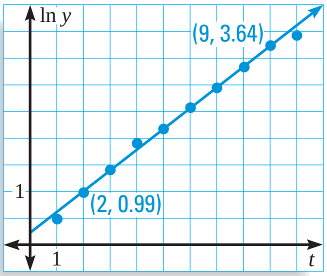
b. Find an exponential model for the original data.

SOLUTION

a. Use a calculator to create a new table of values.

t	1	2	3	4	5	6	7	8	9	10
ln y	0.47	0.99	1.48	1.86	2.19	2.57	2.96	3.34	3.64	3.89

Then plot the new points as shown. The points lie close to a line, so an exponential model should be a good fit for the original data.



b. To find an exponential model $y = ab^t$, choose two points on the line, such as (2, 0.99) and (9, 3.64). Use these points to find an equation of the line. Then solve for y.

$\ln y = 0.379t + 0.233$ **Equation of line**

$y = e^{0.379t + 0.233}$ **Exponentiate each side using base *e*.**

$y = e^{0.233}\left(e^{0.379}\right)^t$ **Use properties of exponents.**

$y = 1.30(1.46)^t$ **Exponential model**

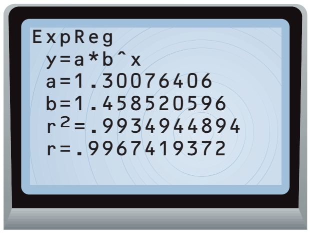
A graphing calculator that performs exponential regression does essentially what is done in Example 2, but uses all of the original data.

REAL LIFE
Communications

EXAMPLE 3 *Using Exponential Regression*

Use a graphing calculator to find an exponential model for the data in Example 2. Use the model to estimate the number of cell-phone subscribers in 1998.

SOLUTION



Enter the original data into a graphing calculator and perform an exponential regression. The model is:

$$y = 1.30(1.46)^t$$

Substituting $t = 11$ (for 1998) into the model gives $y = 1.30(1.46)^{11} \approx 84$ million cell-phone subscribers.

GOAL 2 MODELING WITH POWER FUNCTIONS

Recall from Lesson 7.3 that a power function has the form $y = ax^b$. Because there are only two constants (a and b), only two points are needed to determine a power curve through the points.

EXAMPLE 4 *Writing a Power Function*

Write a power function $y = ax^b$ whose graph passes through (2, 5) and (6, 9).

SOLUTION

Substitute the coordinates of the two given points into $y = ax^b$ to obtain two equations in a and b.

$5 = a \cdot 2^b$ **Substitute 5 for *y* and 2 for *x*.**

$9 = a \cdot 6^b$ **Substitute 9 for *y* and 6 for *x*.**

To solve the system, solve for a in the first equation to get $a = \frac{5}{2^b}$, then substitute into the second equation.

$9 = \left(\frac{5}{2^b}\right)6^b$ **Substitute $\frac{5}{2^b}$ for *a*.**

$9 = 5 \cdot 3^b$ **Simplify.**

$1.8 = 3^b$ **Divide each side by 5.**

$\log_3 1.8 = b$ **Take $\log_3$ of each side.**

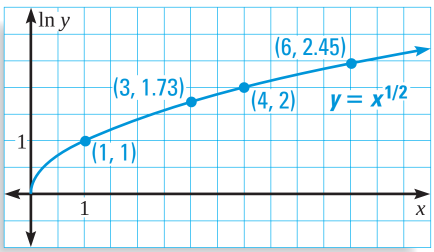
$\frac{\log 1.8}{\log 3} = b$ **Use the change-of-base formula.**

$0.535 \approx b$ **Use a calculator.**

Using $b = 0.535$, you then have $a = \frac{5}{2^b} = \frac{5}{2^{0.535}} \approx 3.45$. So, $y = 3.45x^{0.535}$.

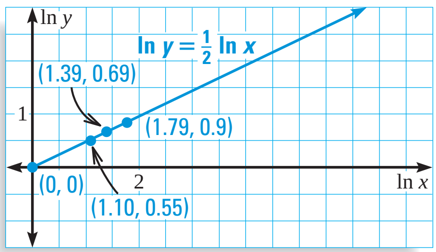
When you are given more than two points, you can decide whether a power model fits the points by plotting the natural logarithms of the y-values against the natural logarithms of the x-values. If the new points $(\ln x, \ln y)$ fit a linear pattern, then the original points (x, y) fit a power pattern.

Graph of points (x, y)



The graph is a power curve.

Graph of points $(\ln x, \ln y)$



The graph is a line.

Real Life Astronomy

EXAMPLE 5 *Finding a Power Model*

The table gives the mean distance x from the sun (in astronomical units) and the period y (in Earth years) of the six planets closest to the sun.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
x	0.387	0.723	1.000	1.524	5.203	9.539
y	0.241	0.615	1.000	1.881	11.862	29.458

a. Draw a scatter plot of ln y versus ln x. Is a power model a good fit for the original data?

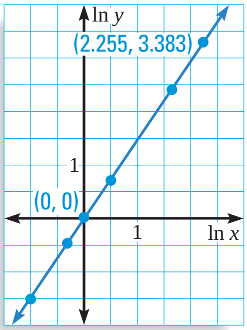
b. Find a power model for the original data.

Solution

a. Use a calculator to create a new table of values.

ln x	-0.949	-0.324	0.000	0.421	1.649	2.255
ln y	-1.423	-0.486	0.000	0.632	2.473	3.383

Then plot the new points, as shown at the right. The points lie close to a line, so a power model should be a good fit for the original data.



b. To find a power model $y = ax^b$, choose two points on the line, such as (0, 0) and (2.255, 3.383). Use these points to find an equation of the line. Then solve for y.

$\ln y = 1.5 \ln x$ — **Equation of line**

$\ln y = \ln x^{1.5}$ — **Power property of logarithms**

$y = x^{1.5}$ — **$\log_b x = \log_b y$ if and only if $x = y$.**

A graphing calculator that performs power regression does essentially what is done in Example 5, but uses all of the original data.

EXAMPLE 6 *Using Power Regression*

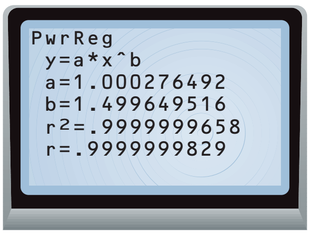
ASTRONOMY Use a graphing calculator to find a power model for the data in Example 5. Use the model to estimate the period of Neptune, which has a mean distance from the sun of 30.043 astronomical units.

Solution

Enter the original data into a graphing calculator and perform a power regression. The model is:

$$y = x^{1.5}$$

Substituting 30.043 for x in the model gives $y = (30.043)^{1.5} \approx 165$ years for the period of Neptune.



Focus on People

JOHANNES KEPLER, a German astronomer and mathematician, was the first person to observe that a planet's distance from the sun and its period were related by the power function in Examples 5 and 6.

Guided Practice

Vocabulary Check ✓

1. Complete this statement: When you are given more than two points, you can decide whether you can fit a(n) _?_ model to the points by plotting the natural logarithms of the y-values against the x-values.

Concept Check ✓

2. How many points determine an exponential function $y = ab^x$? How many points determine a power function $y = ax^b$?

3. Can you use the procedure in Example 5 to find a power model for a data set where one of the points has an x-coordinate of 0? Explain why or why not.

Skill Check ✓

Write an exponential function of the form $y = ab^x$ whose graph passes through the given points.

4. (1, 3), (2, 36) **5.** (2, 2), (4, 18) **6.** (1, 4), (3, 16)

7. (2, 3.5), (1, 5.2) **8.** (5, 8), (3, 32) **9.** $\left(1, \frac{1}{2}\right), \left(3, \frac{3}{8}\right)$

Write a power function of the form $y = ax^b$ whose graph passes through the given points.

10. (3, 27), (9, 243) **11.** (1, 2), (4, 32) **12.** (4, 48), (2, 6)

13. (1, 4), (3, 8) **14.** (4.5, 9.2), (1, 6.4) **15.** $\left(2, \frac{1}{2}\right), \left(4, \frac{3}{5}\right)$

16. **CELL-PHONE USERS** Use the model in Example 3 to estimate the number of cell-phone users in 2005. What does your answer tell you about the model?

Practice and Applications

STUDENT HELP

Extra Practice to help you master skills is on p. 951.

WRITING EXPONENTIAL FUNCTIONS **Write an exponential function of the form $y = ab^x$ whose graph passes through the given points.**

17. (1, 4), (2, 12) **18.** (2, 18), (3, 108) **19.** (6, 8), (7, 32)

20. (1, 7), (3, 63) **21.** (3, 8), (6, 64) **22.** (−3, 3), (4, 6561)

23. $\left(4, \frac{112}{81}\right), \left(-1, \frac{21}{2}\right)$ **24.** (3, 13.5), (5, 30.375) **25.** $\left(2, \frac{25}{4}\right), \left(4, \frac{625}{4}\right)$

FINDING EXPONENTIAL MODELS **Use the table of values to draw a scatter plot of ln y versus x. Then find an exponential model for the data.**

26.

x	1	2	3	4	5	6	7	8
y	14	28	56	112	224	448	896	1792

27.

x	1	2	3	4	5	6	7	8
y	10.2	30.5	43.4	61.2	89.7	120.6	210.4	302.5

28.

x	2	4	6	8	10	12	14	16
y	12.8	20.48	32.77	52.43	83.89	134.22	214.75	343.6

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 17–25
Example 2: Exs. 26–28
Example 3: Exs. 54–56
Example 4: Exs. 29–37
Example 5: Exs. 38–40
Example 6: Exs. 57, 58

WRITING POWER FUNCTIONS **Write a power function of the form $y = ax^b$ whose graph passes through the given points.**

29. (2, 1), (6, 5) **30.** (6, 8), (12, 36) **31.** (5, 12), (7, 25)

32. (3, 4), (6, 18) **33.** (2, 10), (8, 25) **34.** (6, 11), (24, 72)

35. (2.2, 10.4), (8.8, 20.3) **36.** (2.9, 9.4), (7.3, 12.8) **37.** (2.71, 6.42), (13.55, 29.79)

FINDING POWER MODELS **Use the table of values to draw a scatter plot of ln y versus ln x. Then find a power model for the data.**

38.

x	1	2	3	4	5	6	7
y	0.78	7.37	27.41	69.63	143.47	259.00	426.79

39.

x	1	2	3	4	5	6	7
y	1.2	5.4	9.8	14.3	25.6	41.2	65.8

40.

x	2	4	6	8	10	12	14
y	1.89	1.44	1.22	1.09	1.00	0.93	0.87

WRITING EQUATIONS **Write y as a function of x.**

41. $\log y = 0.24x + 4.5$

42. $\log y = 0.2 \log x + 0.8$

43. $\ln y = x + 4$

44. $\log y = -0.12 + 0.88x$

45. $\log y = -0.48 \log x - 0.548$

46. $\ln y = 2.3 \ln x + 4.7$

47. $\ln y = -2.38x + 0.98$

48. $\log y = -1.48 + 3.751 \log x$

49. $\ln y = -1.5x + 2.5$

50. $1.2 \log y = 3.4 \log x$

51. $\frac{1}{2} \log y = \frac{5}{6} \log x$

52. $2\frac{1}{8} \ln y = 4\frac{1}{4} \ln x + \frac{3}{8}$

53. VISUAL THINKING Find equations of the line, the exponential curve, and the power curve that each pass through the points (1, 3) and (2, 12). Graph the equations in the same coordinate plane and then describe what happens when the equations are used as models to predict y-values for x-values greater than 2.

MODELING DATA **In Exercises 54–58, you may wish to use a graphing calculator to perform exponential regression or power regression.**

54. NEW WEB SITE You have just created your own Web site. You are keeping track of the number of hits (the number of visits to the site). The table shows the number y of hits in each of the first 10 months where x is the month number.

x	1	2	3	4	5	6	7	8	9	10
y	22	39	70	126	227	408	735	1322	2380	4285

a. Find an exponential model for the data.

b. According to your model, how many hits do you expect in the twelfth month?

c. According to your model, how many hits would there be in the thirty-fourth month? What is wrong with this number?

FOCUS ON APPLICATIONS

REAL LIFE **CRANES** The red-crowned crane (*Grus japonensis*) is the second-rarest crane species, with a total population in the wild of about 1700–2000 birds.

55. **CRANES** The table shows the number C of cranes in Izumi, Japan, from 1950 to 1990 where t represents the number of years since 1950.

▶ Source: Yamashina Institute of Ornithology

t	0	5	10	15	20	25	30	35	40
C	293	299	438	1573	2336	3649	5602	7610	9959

a. Draw a scatter plot of $\ln C$ versus t. Is an exponential model a good fit for the original data?

b. Find an exponential model for the original data. Estimate the number of cranes in Izumi, Japan, in the year 2000.

56. **UNITED STATES STAMPS** The table shows the cumulative number s of different stamps in the United States from 1889 to 1989 where t represents the number of years since 1889.

t	0	10	20	30	40	50	60	70	80	90	100
s	218	293	374	541	681	858	986	1138	1138	1794	2438

a. Draw a scatter plot of $\ln s$ versus t. Is an exponential model a good fit for the original data?

b. Find an exponential model for the original data. Estimate the cumulative number of stamps in the United States in the year 2000.

57. **CITIES OF ARGENTINA** The table shows the population y (in millions) and the population rank x for nine cities in Argentina in 1991.

a. Draw a scatter plot of $\ln y$ versus $\ln x$. Is a power model a good fit for the original data?

b. Find a power model for the original data. Estimate the population of the city Vicente López, which has a population rank of 20.

City	Rank, x	Population (millions), y
Cordoba	2	1.21
Rosario	3	1.12
La Matanza	4	1.11
Mendoza	5	0.77
La Plata	6	0.64
Moron	7	0.64
San Miguel de Tucuman	8	0.62
Lomas de Zamoras	9	0.57
Mar de Plata	10	0.51

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with Ex. 57.

58. SCIENCE CONNECTION The table shows the atomic number x and the melting point y (in degrees Celsius) for the alkali metals.

Alkali metal	Lithium	Sodium	Potassium	Rubidium	Cesium
Atomic number, x	3	11	19	37	55
Melting point, y	180.5	97.8	63.7	38.9	28.5

a. Draw a scatter plot of $\ln y$ versus $\ln x$. Is a power model a good fit for the original data?

b. Find a power model for the original data.

c. One of the alkali metals, francium, is not shown in the table. It has an atomic number of 87. Using your model, predict the melting point of francium.

Test Preparation

59. MULTI-STEP PROBLEM The femur is a large bone found in the leg or hind limb of an animal. Scientists use the circumference of an animal's femur to estimate the animal's weight. The table at the right shows the femur circumference C (in millimeters) and the weight W (in kilograms) of several animals.

Animal	C (mm)	W (kg)
Meadow mouse	5.5	0.047
Guinea pig	15	0.385
Otter	28	9.68
Cheetah	68.7	38
Warthog	72	90.5
Nyala	97	134.5
Grizzly bear	106.5	256
Kudu	135	301
Giraffe	173	710

▶ Source: Zoological Society of London

a. Draw two scatter plots, one of $\ln W$ versus C and another of $\ln W$ versus $\ln C$.

b. *Writing* Looking at your scatter plots, tell which type of model you think is a better fit for the original data. Explain your reasoning.

c. Using your answer from part (b), find a model for the original data.

d. The table at the right shows the femur circumference C (in millimeters) of four animals. Use the model you found in part (c) to estimate the weight of each animal.

Animal	C (mm)
Raccoon	28
Cougar	60.25
Bison	167.5
Hippopotamus	208

★ **Challenge**

60. DERIVING FORMULAS Using $y = ab^x$ and $y = ax^b$, take the natural logarithm of both sides of each equation. What is the slope and y-intercept of the line relating x and $\ln y$ for $y = ab^x$? of the line relating $\ln x$ and $\ln y$ for $y = ax^b$?

MIXED REVIEW

DESCRIBING END BEHAVIOR **Describe the end behavior of the graph of the polynomial function by completing the statements $f(x) \to$ _?_ as $x \to -\infty$ and $f(x) \to$ _?_ as $x \to +\infty$.** (Review 6.2 for 8.8)

61. $f(x) = -x^3 + x^2 - x + 4$

62. $f(x) = x^4 - 7x^2 + 2$

63. $f(x) = -x^4 + 3x - 3$

64. $f(x) = 3x^5 - x^4 - x^2 + 1$

65. $f(x) = x^6 - 2x - 1$

66. $f(x) = -2x^5 + 3x^4 - 2x^3 + x^2 + 5$

GRAPHING FUNCTIONS **Graph the function.** (Review 8.3 for 8.8)

67. $y = 4e^{-0.75x}$

68. $y = 10e^{-0.4x}$

69. $y = 2e^{x-3}$

70. $y = e^{0.5x} + 2$

71. $y = e^{-0.25x} - 4$

72. $y = 3e^{-1.5x} - 1$

73. $y = 2e^{0.25x} + 1$

74. $y = e^{x+1} - 5$

75. $y = 2.5e^{-0.6x} + 2$

CONDENSING EXPRESSIONS **Condense the expression.** (Review 8.5)

76. $5 \log 2 - \log 8$

77. $2 \log 9 - \log 3$

78. $\ln x + 5 \ln 3$

79. $2 \ln x - \ln 4$

80. $\log_2 8 + 3 \log_2 3 - \log_2 6$

81. $\log_7 12 + 3 \log_7 4 + \log_7 5$