8.1 Exponential Growth

What *you should learn*

GOAL 1 Graph exponential growth functions.

GOAL 2 Use exponential growth functions to model **real-life** situations, such as Internet growth in **Example 3**.

Why *you should learn it*

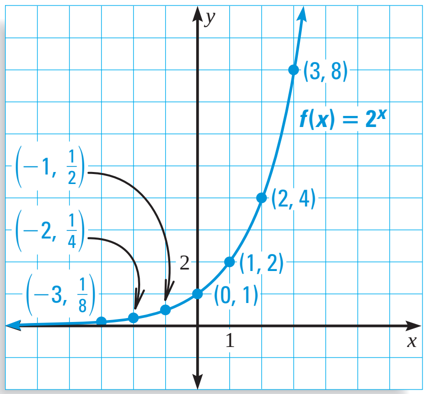
▼ To solve **real-life** problems, such as finding the amount of energy generated from wind turbines in **Exs. 49–51**.

REAL LIFE

GOAL 1 GRAPHING EXPONENTIAL GROWTH FUNCTIONS

An **exponential function** involves the expression b^x where the **base** b is a positive number other than 1. In this lesson you will study exponential functions for which $b > 1$. To see the basic shape of the graph of an exponential function such as $f(x) = 2^x$, you can make a table of values and plot points, as shown below.

x	$f(x) = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



Notice the end behavior of the graph. As $x \to +\infty$, $f(x) \to +\infty$, which means that the graph moves up to the right. As $x \to -\infty$, $f(x) \to 0$, which means that the graph has the line $y = 0$ as an *asymptote*. An **asymptote** is a line that a graph approaches as you move away from the origin.

ACTIVITY Developing Concepts

Investigating Graphs of Exponential Functions

1. Graph $y = \frac{1}{3} \cdot 2^x$ and $y = 3 \cdot 2^x$. Compare the graphs with the graph of $y = 2^x$.

2. Graph $y = -\frac{1}{5} \cdot 2^x$ and $y = -5 \cdot 2^x$. Compare the graphs with the graph of $y = 2^x$.

3. Describe the effect of a on the graph of $y = a \cdot 2^x$ when a is positive and when a is negative.

In the activity you may have observed the following about the graph of $y = a \cdot 2^x$:

- The graph passes through the point $(0, a)$. That is, the y-intercept is a.
- The x-axis is an asymptote of the graph.
- The domain is all real numbers.
- The range is $y > 0$ if $a > 0$ and $y < 0$ if $a < 0$.

The characteristics of the graph of $y = a \cdot 2^x$ listed on the previous page are true of the graph of $y = ab^x$. If $a > 0$ and $b > 1$, the function $y = ab^x$ is an **exponential growth function**.

EXAMPLE 1 *Graphing Exponential Functions of the Form $y = ab^x$*

> **STUDENT HELP**
>
> **Look Back**
> For help with end behavior of graphs, see p. 331.

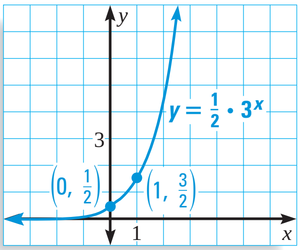
Graph the function.

a. $y = \frac{1}{2} \cdot 3^x$

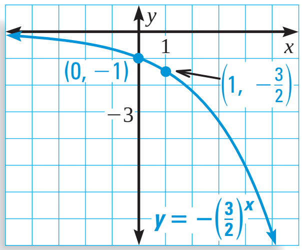
b. $y = -\left(\frac{3}{2}\right)^x$

SOLUTION

a. Plot $\left(0, \frac{1}{2}\right)$ and $\left(1, \frac{3}{2}\right)$. Then, from left to right, draw a curve that begins just above the x-axis, passes through the two points, and moves up to the right.



b. Plot $(0, -1)$ and $\left(1, -\frac{3}{2}\right)$. Then, from left to right, draw a curve that begins just below the x-axis, passes through the two points, and moves down to the right.



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To graph a general exponential function,

$$y = ab^{x-h} + k,$$

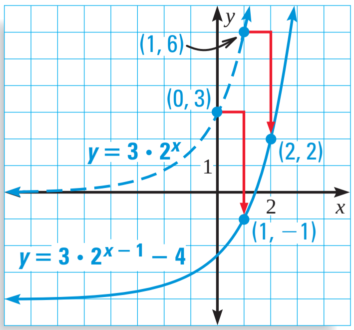
begin by sketching the graph of $y = ab^x$. Then translate the graph horizontally by h units and vertically by k units.

EXAMPLE 2 *Graphing a General Exponential Function*

Graph $y = 3 \cdot 2^{x-1} - 4$. State the domain and range.

SOLUTION

Begin by lightly sketching the graph of $y = 3 \cdot 2^x$, which passes through $(0, 3)$ and $(1, 6)$. Then translate the graph 1 unit to the right and 4 units down. Notice that the graph passes through $(1, -1)$ and $(2, 2)$. The graph's asymptote is the line $y = -4$. The domain is all real numbers, and the range is $y > -4$.



GOAL 2 USING EXPONENTIAL GROWTH MODELS

When a real-life quantity increases by a fixed percent each year (or other time period), the amount y of the quantity after t years can be modeled by this equation:

$$y = a(1 + r)^t$$

In this model, a is the initial amount and r is the percent increase expressed as a decimal. The quantity $1 + r$ is called the **growth factor**.

EXAMPLE 3 *Modeling Exponential Growth*

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

INTERNET HOSTS In January, 1993, there were about 1,313,000 Internet hosts. During the next five years, the number of hosts increased by about 100% per year.
▶ Source: Network Wizards

a. Write a model giving the number h (in millions) of hosts t years after 1993. About how many hosts were there in 1996?

b. Graph the model.

c. Use the graph to estimate the year when there were 30 million hosts.

SOLUTION

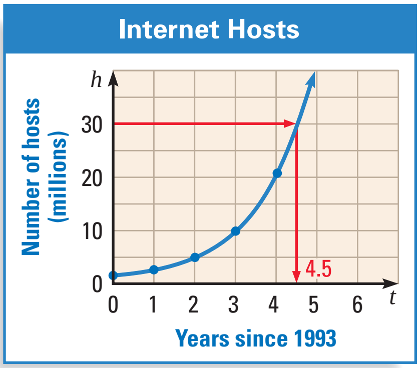
a. The initial amount is $a = 1.313$ and the percent increase is $r = 1$. So, the exponential growth model is:

$$h = a(1 + r)^t \quad \text{Write exponential growth model.}$$
$$= 1.313(1 + 1)^t \quad \text{Substitute for } a \text{ and } r.$$
$$= 1.313 \cdot 2^t \quad \text{Simplify.}$$

Using this model, you can estimate the number of hosts in 1996 ($t = 3$) to be $h = 1.313 \cdot 2^3 \approx 10.5$ million.

b. The graph passes through the points (0, 1.313) and (1, 2.626). It has the t-axis as an asymptote. To make an accurate graph, plot a few other points. Then draw a smooth curve through the points.

c. Using the graph, you can estimate that the number of hosts was 30 million sometime during 1997 ($t \approx 4.5$).



FOCUS ON APPLICATIONS

INTERNET HOSTS A *host* is a computer that stores information you can access through the Internet. For example, Web sites are stored on host computers.

APPLICATION LINK www.mcdougallittell.com

In Example 3 notice that the annual percent increase was 100%. This translated into a growth factor of 2, which means that the number of Internet hosts doubled each year.

People often confuse percent increase and growth factor, especially when a percent increase is 100% or more. For example, a percent increase of 200% means that a quantity *tripled*, because the growth factor is $1 + 2 = 3$. When you hear or read reports of how a quantity has changed, be sure to pay attention to whether a percent increase or a growth factor is being discussed.

COMPOUND INTEREST Exponential growth functions are used in real-life situations involving *compound interest.* Compound interest is interest paid on the initial investment, called the *principal,* and on previously earned interest. (Interest paid only on the principal is called *simple interest*.)

Although interest earned is expressed as an *annual* percent, the interest is usually compounded more frequently than once per year. Therefore, the formula $y = a(1 + r)^t$ must be modified for compound interest problems.

COMPOUND INTEREST

Consider an initial principal P deposited in an account that pays interest at an annual rate r (expressed as a decimal), compounded n times per year. The amount A in the account after t years can be modeled by this equation:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

EXAMPLE 4 *Finding the Balance in an Account*

FINANCE You deposit $1000 in an account that pays 8% annual interest. Find the balance after 1 year if the interest is compounded with the given frequency.

a. annually **b.** quarterly **c.** daily

SOLUTION

a. With interest compounded annually, the balance at the end of 1 year is:

$$A = 1000\left(1 + \frac{0.08}{1}\right)^{1 \cdot 1} \qquad P = 1000,\ r = 0.08,\ n = 1,\ t = 1$$

$$= 1000(1.08)^1 \qquad \text{Simplify.}$$

$$= 1080 \qquad \text{Use a calculator.}$$

▶ The balance at the end of 1 year is $1080.

b. With interest compounded quarterly, the balance at the end of 1 year is:

$$A = 1000\left(1 + \frac{0.08}{4}\right)^{4 \cdot 1} \qquad P = 1000,\ r = 0.08,\ n = 4,\ t = 1$$

$$= 1000(1.02)^4 \qquad \text{Simplify.}$$

$$\approx 1082.43 \qquad \text{Use a calculator.}$$

▶ The balance at the end of 1 year is $1082.43.

c. With interest compounded daily, the balance at the end of 1 year is:

$$A = 1000\left(1 + \frac{0.08}{365}\right)^{365 \cdot 1} \qquad P = 1000,\ r = 0.08,\ n = 365,\ t = 1$$

$$\approx 1000(1.000219)^{365} \qquad \text{Simplify.}$$

$$\approx 1083.28 \qquad \text{Use a calculator.}$$

▶ The balance at the end of 1 year is $1083.28.

FOCUS ON CAREERS

REAL LIFE **FINANCIAL PLANNER**

Financial planners interview clients to determine their assets, liabilities, and financial objectives. They analyze this information and develop an individual financial plan.

INTERNET **CAREER LINK**
www.mcdougallittell.com

GUIDED PRACTICE

Vocabulary Check ✓

1. What is an asymptote?

Concept Check ✓

2. Given the general exponential function $f(x) = ab^{x-h} + k$, describe the effects of a, h, and k on the graph of the function.

3. For what values of b does $y = b^x$ represent exponential growth?

Skill Check ✓

Graph the function. State the domain and range.

4. $y = 4^x$
5. $y = 3^{x-1}$
6. $y = 2^{x+2}$
7. $y = 5^x - 3$
8. $y = 5^{x+1} + 2$
9. $y = 2^{x-3} + 1$

10. What is the asymptote of the graph of $y = 3 \cdot 4^{x-1} + 2$? What is the value of y when $x = 2$?

11. **POPULATION** The population of Winnemucca, Nevada, can be modeled by $P = 6191(1.04)^t$ where t is the number of years since 1990. What was the population in 1990? By what percent did the population increase each year?

12. **ACCOUNT BALANCE** You deposit \$500 in an account that pays 3% annual interest. Find the balance after 2 years if the interest is compounded with the given frequency.

   a. annually **b.** quarterly **c.** daily

PRACTICE AND APPLICATIONS

STUDENT HELP

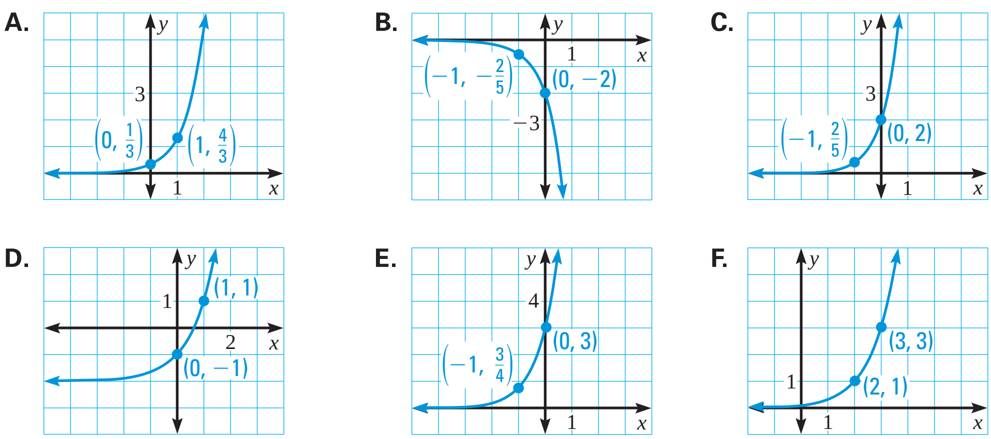
Extra Practice to help you master skills is on p. 950.

INVESTIGATING GRAPHS **Identify the *y*-intercept and the asymptote of the graph of the function.**

13. $y = 5^x$
14. $y = -2 \cdot 4^x$
15. $y = 4 \cdot 2^x$
16. $y = 2^x - 1$
17. $y = 3 \cdot 2^{x-1}$
18. $y = 2 \cdot 3^{x-4}$

MATCHING GRAPHS **Match the function with its graph.**

19. $y = 2 \cdot 5^x$
20. $y = 3 \cdot 4^x$
21. $y = -2 \cdot 5^x$
22. $y = \frac{1}{3} \cdot 4^x$
23. $y = 3^{x-2}$
24. $y = 3^x - 2$



STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 13–15, 19–22, 25–33
Example 2: Exs. 16–18, 23, 24, 34–42
Example 3: Exs. 43–54, 56, 58, 66
Example 4: Exs. 55, 57, 59–65, 67

GRAPHING FUNCTIONS **Graph the function.**

25. $y = 5^x$ **26.** $y = -2^x$ **27.** $y = 8 \cdot 2^x$

28. $y = -3 \cdot 2^x$ **29.** $y = -2 \cdot 5^x$ **30.** $y = -(2.5)^x$

31. $y = 6\left(\frac{5}{4}\right)^x$ **32.** $y = -\frac{2}{3} \cdot 3^x$ **33.** $y = -\frac{1}{5}(1.5)^x$

GRAPHING FUNCTIONS **Graph the function. State the domain and range.**

34. $y = -2 \cdot 3^{x+2}$ **35.** $y = 4 \cdot 5^{x-1}$ **36.** $y = 7 \cdot 3^{x-2}$

37. $y = 3 \cdot 4^{x-1}$ **38.** $y = 3^{x+1} + 1$ **39.** $y = 2^{x-3} + 3$

40. $y = -3 \cdot 6^{x+2} - 2$ **41.** $y = 4 \cdot 2^{x-3} + 1$ **42.** $y = 8 \cdot 2^{x-3} - 3$

NATURAL GAS **In Exercises 43–45, use the following information.**
The amount g (in trillions of cubic feet) of natural gas consumed in the United States from 1940 to 1970 can be modeled by

$$g = 2.91(1.07)^t$$

where t is the number of years since 1940. ▶ Source: *Wind Energy Comes of Age*

43. Identify the initial amount, the growth factor, and the annual percent increase.

44. Graph the function.

45. Estimate the natural gas consumption in 1955.

COMPUTER CHIPS **In Exercises 46–48, use the following information.**
From 1971 to 1995, the average number n of transistors on a computer chip can be modeled by

$$n = 2300(1.59)^t$$

where t is the number of years since 1971.

46. Identify the initial amount, the growth factor, and the annual percent increase.

47. Graph the function.

48. Estimate the number of transistors on a computer chip in 1998.

WIND ENERGY **In Exercises 49–51, use the following information.**
In 1980 wind turbines in Europe generated about 5 gigawatt-hours of energy. Over the next 15 years, the amount of energy increased by about 59% per year.

49. Write a model giving the amount E (in gigawatt-hours) of energy t years after 1980. About how much wind energy was generated in 1984?

50. Graph the model.

51. Estimate the year when 80 gigawatt-hours of energy were generated.

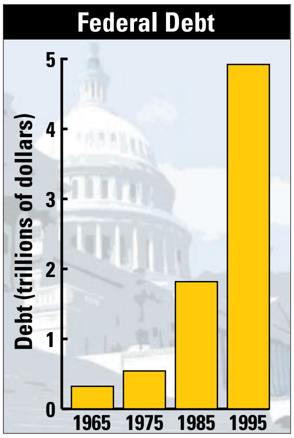
FEDERAL DEBT **In Exercises 52–54, use the following information.**
In 1965 the federal debt of the United States was $322.3 billion. During the next 30 years, the debt increased by about 10.2% each year. ▶ Source: U.S. Bureau of the Census

52. Write a model giving the amount D (in billions of dollars) of debt t years after 1965. About how much was the federal debt in 1980?

53. Graph the model.

54. Estimate the year when the federal debt was $2,120 billion.

FOCUS ON APPLICATIONS



REAL LIFE **FEDERAL DEBT** When the government has an annual deficit it must borrow money. The accumulation of this borrowing is called the Federal debt.

STUDENT HELP

INTERNET KEYSTROKE HELP
Visit our Web site www.mcdougallittell.com to see keystrokes for several models of calculators.

55. **EARNING INTEREST** You deposit $2500 in a bank that pays 4% interest compounded annually. Use the process below and a graphing calculator to determine the balance of your account each year.

a. Enter the initial deposit, 2500, into the calculator. Then enter the formula ANS + ANS × 0.04 to find the balance after one year.

b. What is the balance after five years? (*Hint:* The balance after each year will be displayed each time you press the ENTER key.)

c. How would you enter the formula in part (a) if the interest is compounded quarterly? What do you have to do to find the balance after one year?

d. Find the balance after 5 years if the interest is compounded quarterly. Compare this result with your answer to part (b).

WRITING MODELS **In Exercises 56–58, write an exponential growth model that describes the situation.**

56. **COIN COLLECTING** You buy a commemorative coin for $110. Each year t, the value V of the coin increases by 4%.

57. **SAVINGS ACCOUNT** You deposit $400 in an account that pays 2% annual interest compounded quarterly.

58. **ANTIQUES** You purchase an antique table for $525. Each year t, the value V of the table increases by 5%.

ACCOUNT BALANCE **In Exercises 59–61, use the following information.** You deposit $1600 in a bank account. Find the balance after 3 years for each of the following situations.

59. The account pays 2.5% annual interest compounded monthly.

60. The account pays 1.75% annual interest compounded quarterly.

61. The account pays 4% annual interest compounded yearly.

DEPOSITING FUNDS **In Exercises 62–64, use the following information.** You want to have $2500 after 2 years. Find the amount you should deposit for each of the situations described below.

62. The account pays 2.25% annual interest compounded monthly.

63. The account pays 2% annual interest compounded quarterly.

64. The account pays 5% annual interest compounded yearly.

65. **CRITICAL THINKING** Juan and Michelle each have $800. Juan plans to invest $200 for each of the next four years, while Michelle plans to invest all $800 now. Both accounts pay 3% annual interest compounded monthly. Will they have the same amount of money after four years? If not, explain why.

66. **LAND VALUE** You have inherited land that was purchased for $30,000 in 1960. The value V of the land increased by approximately 5% per year.

a. Write a model for the value of the land t years after 1960.

b. What is the approximate value of the land in the year 2010?

67. **LOGICAL REASONING** Is investing $4000 at 5% annual interest and $4000 at 7% annual interest equivalent to investing $8000 (the total of the two principals) at 6% annual interest (the average of the two interest rates)? Explain.

Test Preparation

68. MULTIPLE CHOICE The student enrollment E of a high school was 1240 in 1990 and increased by 15% per year until 1996. Which exponential growth model shows the school's student enrollment in terms of t, the number of years since 1990?

Ⓐ $E = 15(1240)^t$ Ⓑ $E = 1240(1.15)^t$ Ⓒ $E = 1240(15)^t$

Ⓓ $E = 0.15(1240)^t$ Ⓔ $E = 1.15(1240)^t$

69. MULTIPLE CHOICE Which function is graphed below?

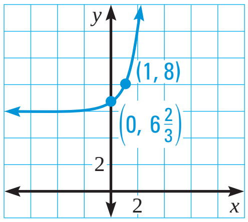
Ⓐ $f(x) = -3^{x+1} + 6$

Ⓑ $f(x) = -2 \cdot 3^{x+1} + 6$

Ⓒ $f(x) = 2 \cdot 3^{x+1} + 6$

Ⓓ $f(x) = 2 \cdot 3^{x-1} + 6$

Ⓔ $f(x) = 3^{x+1} + 6$



★ **Challenge**

70. **IRRATIONAL EXPONENTS** Use a calculator to evaluate the following powers. Round the results to five decimal places.

$3^{14/10}, 3^{141/100}, 3^{1,414/1,000}, 3^{14,142/10,000}, 3^{141,421/100,000}, 3^{1,414,213/1,000,000}$

Each of these powers has a rational exponent. Explain how you can use these powers to define $3^{\sqrt{2}}$, which has an irrational exponent.

EXTRA CHALLENGE
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MIXED REVIEW

EVALUATING POWERS **Evaluate the expression.** **(Review 1.2 for 8.2)**

71. $\left(\frac{1}{2}\right)^3$ **72.** $\left(\frac{3}{7}\right)^3$ **73.** $\left(\frac{1}{2}\right)^5$ **74.** $\left(\frac{5}{8}\right)^4$

75. $\left(\frac{7}{12}\right)^3$ **76.** $\left(\frac{2}{3}\right)^4$ **77.** $\left(\frac{4}{5}\right)^2$ **78.** $\left(\frac{3}{10}\right)^5$

EVALUATING EXPRESSIONS **Evaluate the expression using a calculator. Round the result to two decimal places when appropriate.** **(Review 7.1)**

79. $8^{3/8}$ **80.** $15,625^{1/6}$ **81.** $-243^{1/5}$ **82.** $1024^{1/5}$

83. $10^{1/2}$ **84.** $106^{1/3}$ **85.** $\sqrt[4]{81}$ **86.** $\sqrt[7]{100}$

87. $\sqrt[3]{28}$ **88.** $\sqrt[4]{120}$ **89.** $\sqrt[4]{9}$ **90.** $\sqrt[6]{180}$

OPERATIONS WITH FUNCTIONS **Let $f(x) = 6x - 11$ and $g(x) = 4x^2$. Perform the indicated operation and state the domain.** **(Review 7.3)**

91. $f(x) + g(x)$ **92.** $f(x) - g(x)$ **93.** $f(x) \cdot g(x)$

94. $g(x) - f(x)$ **95.** $f(g(x))$ **96.** $g(f(x))$

97. $\frac{f(x)}{g(x)}$ **98.** $\frac{g(x)}{f(x)}$ **99.** $f(f(x))$

100. **FENCING** You want to build a rectangular pen for your dog using 40 feet of fencing. The area of the pen should be 90 square feet. What should the dimensions of the pen be? **(Review 5.2)**