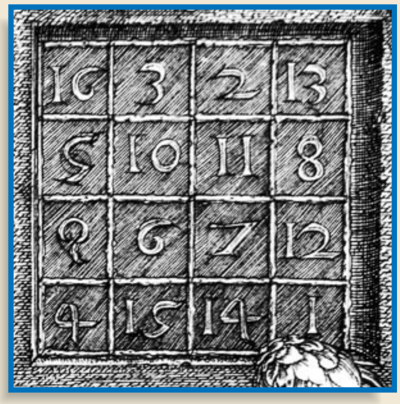
PROJECT
Applying Chapters 4–6

Magic Squares

OBJECTIVE **Explore the mathematics behind magic squares.**

Materials: paper, pencil



A *magic square* is a square array of consecutive integers, usually (but not always) beginning with 1, for which the sum of the entries in each row, column, and diagonal is the same. This common sum is called the *magic constant*.

For example, a 4×4 magic square with a magic constant of 34 is shown. This square appears in the engraving *Melancholia*, which was created in 1514 by the German artist and mathematician Albrecht Dürer.

Magic squares were discovered in China around 2200 B.C. and later spread to India, Japan, and eventually to Europe. The challenge of creating magic squares has fascinated mathematicians and puzzle lovers for many centuries.

HOW TO MAKE A 3 × 3 MAGIC SQUARE

	5	

1 Draw a 3×3 square. You want to use the integers 1 through 9 to fill in the square. Start by writing the middle value, 5, in the center.

4	9	2
3	5	7
8	1	6

2 Continue filling in numbers until you have an arrangement where the entries in each row, column, and diagonal add up to 15.

INVESTIGATION

1. Think of a magic square as a matrix. Suppose a 3×3 matrix containing all 2's is added to the magic square in **Step 2**. Is the resulting matrix also a magic square? If so, what is the magic constant?

2. Generalize your work from Exercise 1 by adding a 3×3 matrix containing all a's, where a represents *any* integer, to the magic square in **Step 2**. Is the result always a magic square? If so, what is the magic constant in terms of a?

3. Use the integers 1 through 9 to make another 3×3 magic square. Add your square to the one in **Step 2**. Is the result a magic square? Explain. (Remember that the square's rows, columns, and diagonals must have the same sum *and* the numbers in the square must be consecutive integers.)

4. Use scalar multiplication to multiply the magic square in **Step 2** by the scalar 2. Is the result a magic square? Explain.

7			14
	13	8	
	3	10	
9			4

INVESTIGATION (*continued*)

5. The *transpose* of a matrix A is a matrix A^T obtained by interchanging the rows and columns of A—the first row of A becomes the first column of A^T, the second row of A becomes the second column of A^T, and so on. Find the transpose of the magic square in **Step 2**. Is the transpose also a magic square?

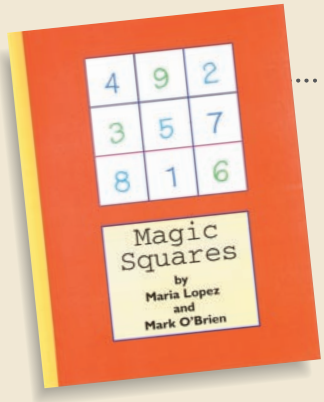
6. Copy and complete the 4×4 magic square shown. What reasoning did you use to place the remaining numbers?

7. The sum S of the first k positive integers is given by the quadratic function $S = \frac{1}{2}k^2 + \frac{1}{2}k$. Use this function to find the sum of the entries in the 3×3 and 4×4 magic squares from **Step 2** and Exercise 6. Check your answers by computing the sums directly.

8. Consider an $n \times n$ magic square that contains the integers 1 through n^2. Use the function from Exercise 7 to write a formula for the sum S of the entries in the square in terms of n. What type of function is this formula?

9. For an $n \times n$ magic square that contains the integers 1 through n^2, write a formula for the square's magic constant M in terms of n. (*Hint:* Note that the magic constant is the sum of all the entries in the square divided by the number of rows or columns.) What type of function is this formula?

PRESENT YOUR RESULTS



Write a report to present your results.

- Include the 3×3 and 4×4 magic squares you made.
- Tell whether a magic square is produced by performing each of the following operations on an $n \times n$ magic square A: adding the same integer to each entry of A, multiplying each entry of A by the same integer, adding another $n \times n$ magic square to A, and taking the transpose of A.
- Include the formulas you found for the sum of the entries and for the magic constant of an $n \times n$ magic square containing the integers 1 through n^2.
- Describe how you used your knowledge of matrices, quadratic functions, and higher-degree polynomial functions in this project.

EXTENSION

Consider an $n \times n$ magic square containing the integers a through $a + n^2 - 1$. Such a magic square is shown at the right for $n = 3$ and $a = 5$. For this type of magic square, write formulas for the sum S of the entries and for the magic constant M in terms of n and a. Verify that your formulas work for the magic square shown.

8	13	6
7	9	11
12	5	10