

Modeling with Polynomial Functions

GOAL 1 USING FINITE DIFFERENCES

You know that two points determine a line and that three points determine a parabola. In Example 1 you will see that four points determine the graph of a cubic function.

EXAMPLE 1 Writing a Cubic Function

Write the cubic function whose graph is shown at the right.

SOLUTION

Use the three given x -intercepts to write the following:

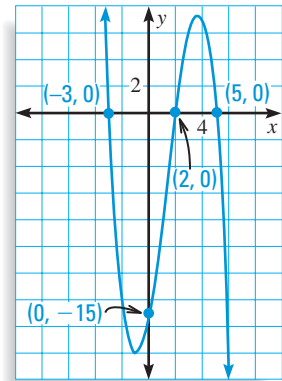
$$f(x) = a(x + 3)(x - 2)(x - 5)$$

To find a , substitute the coordinates of the fourth point.

$$-15 = a(0 + 3)(0 - 2)(0 - 5), \text{ so } a = -\frac{1}{2}$$

$$\blacktriangleright f(x) = -\frac{1}{2}(x + 3)(x - 2)(x - 5)$$

✓CHECK Check the graph's end behavior. The degree of f is odd and $a < 0$, so $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$.



What you should learn

GOAL 1 Use finite differences to determine the degree of a polynomial function that will fit a set of data.

GOAL 2 Use technology to find polynomial models for **real-life** data, as applied in Example 4.

Why you should learn it

▼ To model **real-life** quantities, such as the speed of the space shuttle in Ex. 49.



To decide whether y -values for equally-spaced x -values can be modeled by a polynomial function, you can use **finite differences**.

EXAMPLE 2 Finding Finite Differences

The first three triangular numbers are shown at the right.

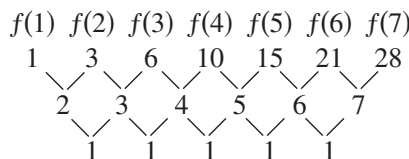
A formula for the n th triangular number is $f(n) = \frac{1}{2}(n^2 + n)$.

Show that this function has constant second-order differences.

SOLUTION

Write the first several triangular numbers. Find the first-order differences by subtracting consecutive triangular numbers.

Then find the second-order differences by subtracting consecutive first-order differences.



Triangular Numbers

• $f(1) = 1$

• • $f(2) = 3$

• • • $f(3) = 6$

Function values for equally-spaced n -values

First-order differences

Second-order differences

In Example 2 notice that the function has degree *two* and that the *second*-order differences are constant. This illustrates the first property of finite differences.


PROPERTIES OF FINITE DIFFERENCES

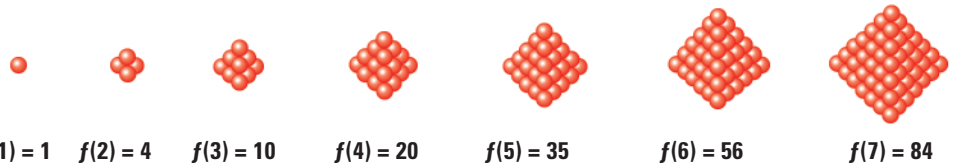
1. If a polynomial function $f(x)$ has degree n , then the n th-order differences of function values for equally spaced x -values are nonzero and constant.
2. Conversely, if the n th-order differences of equally-spaced data are nonzero and constant, then the data can be represented by a polynomial function of degree n .

The following example illustrates the second property of finite differences.

EXAMPLE 3 Modeling with Finite Differences

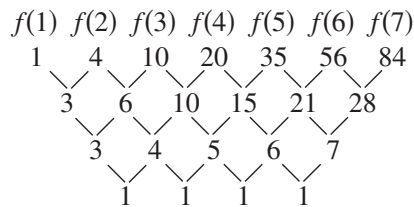
The first six triangular pyramidal numbers are shown below. Find a polynomial function that gives the n th triangular pyramidal number.

STUDENT HELP
 **HOMEWORK HELP**
 Visit our Web site
www.mcdougallittell.com
 for extra examples.



SOLUTION

Begin by finding the finite differences.



Function values for
equally-spaced n -values
 First-order differences
 Second-order differences
 Third-order differences

Because the third-order differences are constant, you know that the numbers can be represented by a cubic function which has the form $f(n) = an^3 + bn^2 + cn + d$.

By substituting the first four triangular pyramidal numbers into the function, you can obtain a system of four linear equations in four variables.

$$\begin{aligned}
 a(1)^3 + b(1)^2 + c(1) + d &= 1 && \longrightarrow && a + b + c + d &= 1 \\
 a(2)^3 + b(2)^2 + c(2) + d &= 4 && \longrightarrow && 8a + 4b + 2c + d &= 4 \\
 a(3)^3 + b(3)^2 + c(3) + d &= 10 && \longrightarrow && 27a + 9b + 3c + d &= 10 \\
 a(4)^3 + b(4)^2 + c(4) + d &= 20 && \longrightarrow && 64a + 16b + 4c + d &= 20
 \end{aligned}$$

Using a calculator to solve the system gives $a = \frac{1}{6}$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, and $d = 0$.

► The n th triangular pyramidal number is given by $f(n) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$.

GOAL 2 POLYNOMIAL MODELING WITH TECHNOLOGY

In Examples 1 and 3 you found a cubic model that *exactly* fits a set of data points. In many real-life situations, you cannot find a simple model to fit data points exactly. Instead you can use the regression feature on a graphing calculator to find an *n*th-degree polynomial model that best fits the data.

EXAMPLE 4 Modeling with Cubic Regression

BOATING The data in the table give the average speed *y* (in knots) of the *Trident* motor yacht for several different engine speeds *x* (in hundreds of revolutions per minute, or RPMs).

- Find a polynomial model for the data.
- Estimate the average speed of the *Trident* for an engine speed of 2400 RPMs.
- What engine speed produces a boat speed of 14 knots?

Engine speed, <i>x</i>	9	11	13	15	17	19	21.5
Boat speed, <i>y</i>	6.43	7.61	8.82	9.86	10.88	12.36	15.24

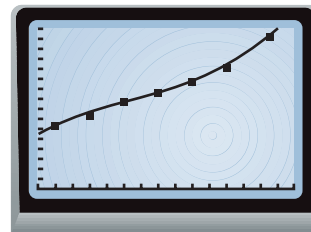
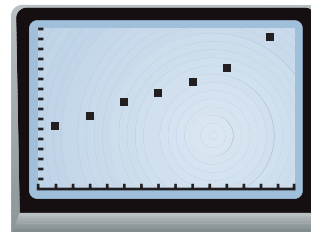
SOLUTION

- Enter** the data in a graphing calculator and make a scatter plot. From the scatter plot, it appears that a cubic function will fit the data better than a linear or quadratic function.

Use cubic regression to obtain a model.

$$y = 0.00475x^3 - 0.194x^2 + 3.13x - 9.53$$

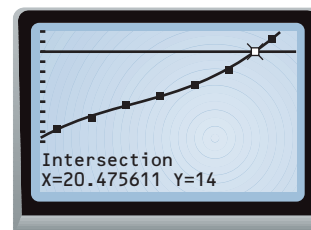
- ✓ **CHECK** By graphing the model in the same viewing window as the scatter plot, you can see that it is a good fit.



- Substitute $x = 24$ into the model from part (a).

$$\begin{aligned} y &= 0.00475(24)^3 - 0.194(24)^2 + 3.13(24) - 9.53 \\ &= 19.51 \end{aligned}$$

- The *Trident*'s speed for an engine speed of 2400 RPMs is about 19.5 knots.
- Graph the model and the equation $y = 14$ on the same screen. Use the *Intersect* feature to find the point where the graphs intersect.
- An engine speed of about 2050 RPMs produces a boat speed of 14 knots.



MOTORBOATS often have tachometers instead of speedometers. The tachometer measures the engine speed in revolutions per minute, which can then be used to determine the speed of the boat.

GUIDED PRACTICE

Vocabulary Check ✓

Concept Check ✓

Skill Check ✓

- Describe what first-order differences and second-order differences are.
- How many points do you need to determine a quartic function?
- Why can't you use finite differences to find a model for the data in Example 4?
- Write the cubic function whose graph passes through $(3, 0)$, $(-1, 0)$, $(-2, 0)$, and $(1, 2)$.

Show that the n th-order finite differences for the given function of degree n are nonzero and constant.

5. $f(x) = 5x^2 - 2x + 1$

6. $f(x) = x^3 + x^2 - 1$

7. $f(x) = x^4 + 2x$

8. $f(x) = 2x^3 - 12x^2 - 5x + 3$

Use finite differences to determine the degree of the polynomial function that will fit the data.

9.

x	1	2	3	4	5	6
$f(x)$	-1	3	3	5	15	39

10.

x	1	2	3	4	5	6
$f(x)$	0	8	12	12	8	0

Find a polynomial function that fits the data.

11.

x	1	2	3	4	5	6
$f(x)$	6	15	22	21	6	-29

12.

x	1	2	3	4	5	6
$f(x)$	-1	-4	-3	8	35	84

13. **GEOMETRY CONNECTION** Find a polynomial function that gives the number of diagonals of a polygon with n sides.

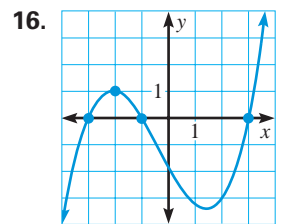
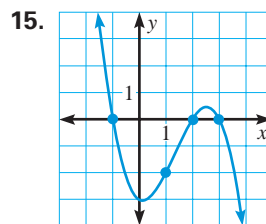
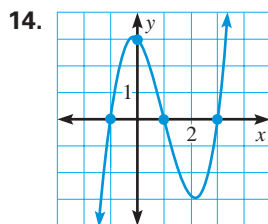
Number of sides, n	3	4	5	6	7	8
Number of diagonals, d	0	2	5	9	14	20

PRACTICE AND APPLICATIONS

STUDENT HELP

→ **Extra Practice** to help you master skills is on p. 949.

WRITING CUBIC FUNCTIONS Write the cubic function whose graph is shown.



FINDING A CUBIC MODEL Write a cubic function whose graph passes through the given points.

17. $(-1, 0)$, $(-2, 0)$, $(0, 0)$, $(1, -3)$

18. $(3, 0)$, $(2, 0)$, $(-3, 0)$, $(1, -1)$

19. $(1, 0)$, $(3, 0)$, $(-2, 0)$, $(2, 1)$

20. $(-1, 0)$, $(-4, 0)$, $(4, 0)$, $(0, 3)$

21. $(3, 0)$, $(2, 0)$, $(-1, 0)$, $(1, 4)$

22. $(0, 0)$, $(-3, 0)$, $(5, 0)$, $(-2, 3)$

STUDENT HELP

→ HOMEWORK HELP

Example 1: Exs. 14–22

Example 2: Exs. 23–31, 44, 45

Example 3: Exs. 32–43, 46

Example 4: Exs. 47–49

FINDING FINITE DIFFERENCES Show that the n th-order differences for the given function of degree n are nonzero and constant.

23. $f(x) = x^2 - 3x + 7$ 24. $f(x) = 2x^3 - 5x^2 - x$ 25. $f(x) = -x^3 + 3x^2 - 2x - 3$

26. $f(x) = x^4 - 3x^3$ 27. $f(x) = 2x^4 - 20x$ 28. $f(x) = -4x^2 + x + 6$

29. $f(x) = -x^4 + 5x^2$ 30. $f(x) = 3x^3 - 5x^2 - 2$ 31. $f(x) = -3x^2 + 4x + 2$

 **FINDING A MODEL** Use finite differences and a system of equations to find a polynomial function that fits the data. You may want to use a calculator.

32.

x	1	2	3	4	5	6
$f(x)$	-4	0	10	26	48	76

33.

x	1	2	3	4	5	6
$f(x)$	17	28	33	32	25	12

34.

x	1	2	3	4	5	6
$f(x)$	-4	-6	-2	14	48	106

35.

x	1	2	3	4	5	6
$f(x)$	-2	-6	-6	4	30	78

36.

x	1	2	3	4	5	6
$f(x)$	-3	-8	-15	-21	-23	-18

37.

x	1	2	3	4	5	6
$f(x)$	2	20	58	122	218	352

38.

x	1	2	3	4	5	6
$f(x)$	-5	0	9	16	15	0

39.

x	1	2	3	4	5	6
$f(x)$	-2	1	-4	-5	10	53

40.

x	1	2	3	4	5	6
$f(x)$	20	-2	-4	2	4	-10

41.

x	1	2	3	4	5	6
$f(x)$	2	-5	-4	-1	-2	-13

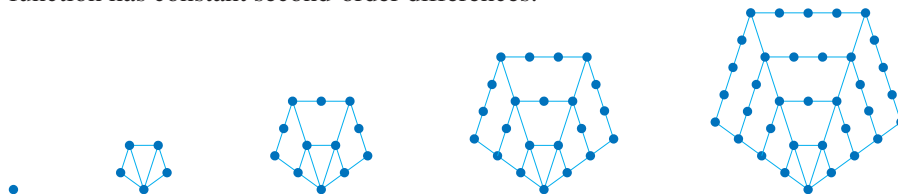
42.

x	1	2	3	4	5	6
$f(x)$	26	-4	-2	2	2	16

43.

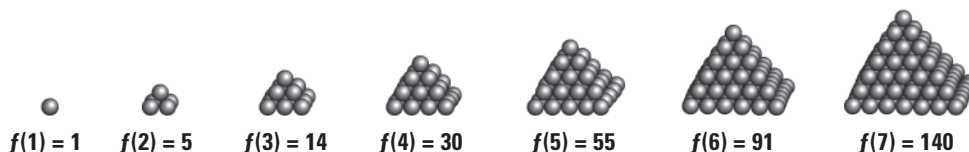
x	1	2	3	4	5	6
$f(x)$	0	6	2	6	12	-10

44. **PENTAGONAL NUMBERS** The dot patterns show pentagonal numbers. A formula for the n th pentagonal number is $f(n) = \frac{1}{2}n(3n - 1)$. Show that this function has constant second-order differences.



45. **HEXAGONAL NUMBERS** A formula for the n th hexagonal number is $f(n) = n(2n - 1)$. Show that this function has constant second-order differences.

46. **SQUARE PYRAMIDAL NUMBERS** The first six square pyramidal numbers are shown. Find a polynomial function that gives the n th square pyramidal number.



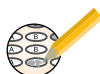
FOCUS ON PEOPLE



EILEEN COLLINS was selected by NASA for the astronaut program in 1990. Since then she has become the first woman to pilot a spacecraft and the first woman to command a space shuttle.



Test Preparation



★ Challenge


EXTRA CHALLENGE

www.mcdougallittell.com

FINDING MODELS In Exercises 47–49, use a graphing calculator to find a polynomial model for the data.

47. **GIRL SCOUTS** The table shows the number of Girl Scouts (in thousands) from 1989 to 1996. Find a polynomial model for the data. Then estimate the number of Girl Scouts in 2000.

t	1989	1990	1991	1992	1993	1994	1995	1996
y	231	253.3	273.8	284.1	294.1	303.6	368.6	383.7

48. **REAL ESTATE** The table shows the average price (in thousands of dollars) of a house in the Northeastern United States for 1987 to 1995. Find a polynomial model for the data. Then predict the average price of a house in the Northeast in 2000.  **DATA UPDATE** of *Statistical Abstract of the United States* data at www.mcdougallittell.com

x	1987	1988	1989	1990	1991	1992	1993	1994	1995
$f(x)$	140	149	159.6	159	155.9	169	162.9	169	180

49. **SPACE EXPLORATION** The table shows the average speed y (in feet per second) of a space shuttle for different times t (in seconds) after launch. Find a polynomial model for the data. When the space shuttle reaches a speed of approximately 4400 feet per second, its booster rockets fall off. Use the model to determine how long after launch this happens.

t	10	20	30	40	50	60	70	80
y	202.4	463.4	748.2	979.3	1186.3	1421.3	1795.4	2283.5

50. **MULTI-STEP PROBLEM** Your friend has a dog-walking service and your cousin has a lawn-care service. You want to start a small business of your own. You are trying to decide which of the two services you should choose. The profits for the first 6 months of the year are shown in the table.

Dog-walking service	Month, t	1	2	3	4	5	6
	Profit, P	3	5	22	54	101	163
Lawn-care service	Month, t	1	2	3	4	5	6
	Profit, P	3	21	41	68	107	163

- a. Use finite differences to find a polynomial model for each business.
- b. **Writing** You want to choose the business that will make the greater profit in December (when $t = 12$). Explain which business you should choose and why.
51. a. Substitute the expressions $x, x + 1, x + 2, \dots, x + 5$ for x in the function $f(x) = ax^3 + bx^2 + cx + d$ and show that third-order differences are constant.
- b. The data below can be modeled by a cubic function. Set the variable expressions you found in part (a) equal to the first-, second-, and third-order differences for these values. Solve the equations to find the coefficients of the function that models the data. Check your work by substituting the original data values into the function.

x	1	2	3	4	5	6
$f(x)$	-1	1	-3	-7	-5	9

MIXED REVIEW

SOLVING QUADRATIC EQUATIONS Solve the equation. (Review 5.3 for 7.1)

52. $3x^2 = 6$

53. $16x^2 = 4$

54. $4x^2 - 5 = 9$

55. $6x^2 + 3 = 16$

56. $-x^2 + 9 = 2x^2 - 6$

57. $-x^2 + 2 = x^2 + 1$

SOLVING EQUATIONS Solve the equation by completing the square. (Review 5.5)

58. $x^2 + 12x + 27 = 0$

59. $x^2 + 6x - 24 = 0$

60. $x^2 - 3x - 18 = 0$

61. $2x^2 + 8x + 11 = 0$

62. $-x^2 + 14x + 15 = 0$

63. $3x^2 - 18x + 32 = 0$

SUM OR DIFFERENCE OF CUBES Factor the polynomial. (Review 6.4)

64. $8x^3 - 1$

65. $27x^3 + 8$

66. $216x^3 + 64$

67. $8x^3 - 125$

68. $3x^3 - 24$

69. $8x^3 + 216$

70. $27x^3 + 1000$

71. $3x^3 + 81$

QUIZ 3

Self-Test for Lessons 6.7–6.9

Find all the zeros of the polynomial function. (Lesson 6.7)

1. $f(x) = 2x^3 - x^2 - 22x - 15$

2. $f(x) = x^3 + 3x^2 + 3x + 2$

3. $f(x) = x^4 - 3x^3 - 2x^2 - 6x - 8$

4. $f(x) = 2x^4 - x^3 - 8x^2 + x + 6$

Write a polynomial of least degree that has real coefficients, the given zeros, and a leading coefficient of 1. (Lesson 6.7)

5. $-2, -2, 2$

6. $0, 1, -3$

7. $4, 2 + i, 2 - i$

8. $2, 5, -i$

9. $4, 2 - 3i$

10. $1 - i, 2 + 2i$

Graph the function. Estimate the local maximums and minimums. (Lesson 6.8)

11. $f(x) = -(x - 2)(x + 3)(x + 1)$

12. $f(x) = x(x - 1)(x + 1)(x + 2)$

13. $f(x) = 2(x - 2)(x - 3)(x - 4)$

14. $f(x) = (x + 1)(x + 3)^2$

Write a cubic function whose graph passes through the points. (Lesson 6.9)

15. $(-2, 0), (2, 0), (-4, 0), (-1, 3)$

16. $(-1, 0), (4, 0), (2, 0), (-3, 1)$

17. $(3, 0), (0, 0), (5, 0), (2, 6)$

18. $(1, 0), (-3, 0), (-5, 0), (-4, 10)$


Find a polynomial function that models the data. (Lesson 6.9)

19.

x	1	2	3	4	5	6
$f(x)$	-5	-6	-1	16	51	110

20.

x	1	2	3	4	5	6
$f(x)$	-1	-4	-3	8	35	84

21.  **SOCIAL SECURITY** The table gives the number of children (in thousands) receiving Social Security for each year from 1988 to 1995. Use a graphing calculator to find a polynomial model for the data. (Lesson 6.9)

Year	1988	1989	1990	1991	1992	1993	1994	1995
Number of children	3204	3165	3187	3268	3391	3527	3654	3734