6.8 Analyzing Graphs of Polynomial Functions

What *you should learn*

GOAL 1 Analyze the graph of a polynomial function.

GOAL 2 Use the graph of a polynomial function to answer questions about **real-life** situations, such as maximizing the volume of a box in **Example 3**.

Why *you should learn it*

▼ To find the maximum and minimum values of **real-life** functions, such as the function modeling orange consumption in the United States in **Ex. 36**.

GOAL 1 ANALYZING POLYNOMIAL GRAPHS

In this chapter you have learned that zeros, factors, solutions, and x-intercepts are closely related concepts. The relationships are summarized below.

CONCEPT SUMMARY — ZEROS, FACTORS, SOLUTIONS, AND INTERCEPTS

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial function. The following statements are equivalent.

ZERO: k is a zero of the polynomial function f.

FACTOR: $x - k$ is a factor of the polynomial $f(x)$.

SOLUTION: k is a solution of the polynomial equation $f(x) = 0$.

If k is a real number, then the following is also equivalent.

X-INTERCEPT: k is an x-intercept of the graph of the polynomial function f.

EXAMPLE 1 *Using x-Intercepts to Graph a Polynomial Function*

Graph the function $f(x) = \frac{1}{4}(x + 2)(x - 1)^2$.

SOLUTION

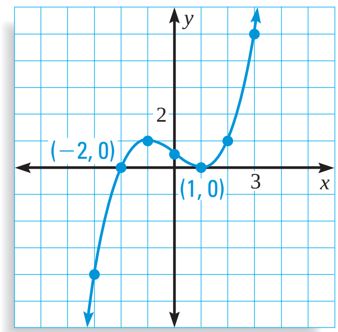
Plot x-intercepts. Since $x + 2$ and $x - 1$ are factors of $f(x)$, -2 and 1 are the x-intercepts of the graph of f. Plot the points $(-2, 0)$ and $(1, 0)$.

Plot points between and beyond the x-intercepts.

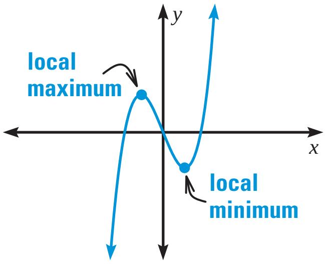
x	-4	-3	-1	0	2	3
y	$-12\frac{1}{2}$	-4	1	$\frac{1}{2}$	1	5

Determine the end behavior of the graph. Because $f(x)$ has three linear factors of the form $x - k$ and a constant factor of $\frac{1}{4}$, it is a cubic function with a positive leading coefficient. Therefore, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

Draw the graph so that it passes through the points you plotted and has the appropriate end behavior.



TURNING POINTS Another important characteristic of graphs of polynomial functions is that they have *turning points* corresponding to local maximum and minimum values. The y-coordinate of a turning point is a **local maximum** of the function if the point is higher than all nearby points. The y-coordinate of a turning point is a **local minimum** if the point is lower than all nearby points.



TURNING POINTS OF POLYNOMIAL FUNCTIONS

The graph of every polynomial function of degree n has *at most* $n - 1$ turning points. Moreover, if a polynomial function has n distinct real zeros, then its graph has *exactly* $n - 1$ turning points.

Recall that in Chapter 5 you used technology to find the maximums and minimums of quadratic functions. In Example 2 you will use technology to find turning points of higher-degree polynomial functions. If you take calculus, you will learn symbolic techniques for finding maximums and minimums.

EXAMPLE 2 *Finding Turning Points*

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

Graph each function. Identify the x-intercepts and the points where the local maximums and local minimums occur.

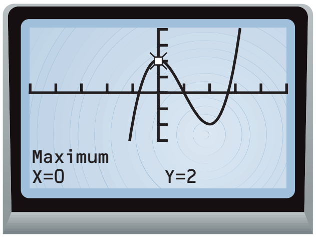
a. $f(x) = x^3 - 3x^2 + 2$

b. $f(x) = x^4 - 4x^3 - x^2 + 12x - 2$

SOLUTION

a. Use a graphing calculator to graph the function.

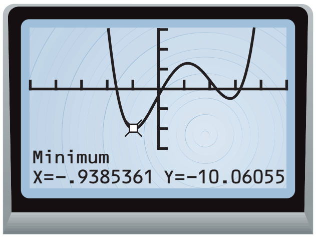
Notice that the graph has three x-intercepts and two turning points. You can use the graphing calculator's *Zero*, *Maximum*, and *Minimum* features to approximate the coordinates of the points.



▸ The x-intercepts of the graph are $x \approx -0.73$, $x = 1$, and $x \approx 2.73$. The function has a local maximum at $(0, 2)$ and a local minimum at $(2, -2)$.

b. Use a graphing calculator to graph the function.

Notice that the graph has four x-intercepts and three turning points. You can use the graphing calculator's *Zero*, *Maximum*, and *Minimum* features to approximate the coordinates of the points.



▸ The x-intercepts of the graph are $x \approx -1.63$, $x \approx 0.17$, $x \approx 2.25$, and $x \approx 3.20$. The function has local minimums at $(-0.94, -10.06)$ and $(2.79, -2.58)$, and it has a local maximum at $(1.14, 6.14)$.

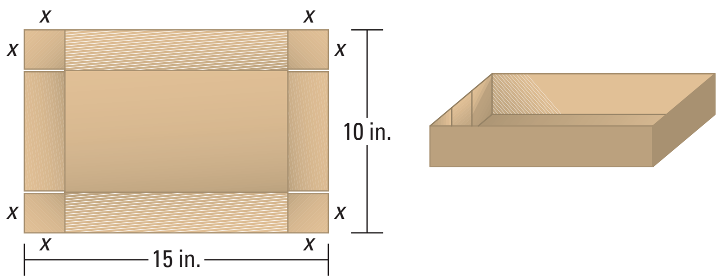
GOAL 2 USING POLYNOMIAL FUNCTIONS IN REAL LIFE

In the following example, technology is used to maximize a polynomial function that models a real-life situation.

REAL LIFE
Manufacturing

EXAMPLE 3 *Maximizing a Polynomial Model*

You are designing an open box to be made of a piece of cardboard that is 10 inches by 15 inches. The box will be formed by making the cuts shown in the diagram and folding up the sides so that the flaps are square. You want the box to have the greatest volume possible. How long should you make the cuts? What is the maximum volume? What will the dimensions of the finished box be?



SOLUTION

PROBLEM SOLVING STRATEGY

VERBAL MODEL Volume = Width · Length · Height

LABELS

Volume = V (cubic inches)

Width = $10 - 2x$ (inches)

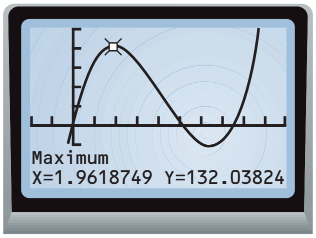
Length = $15 - 2x$ (inches)

Height = x (inches)

ALGEBRAIC MODEL

$$V = (10 - 2x)(15 - 2x)x$$
$$= (4x^2 - 50x + 150)x$$
$$= 4x^3 - 50x^2 + 150x$$

To find the maximum volume, graph the volume function on a graphing calculator as shown at the right. When you use the *Maximum* feature, you consider only the interval $0 < x < 5$ because this describes the physical restrictions on the size of the flaps. From the graph, you can see that the maximum volume is about 132 and occurs when $x \approx 1.96$.



▶ You should make the cuts approximately 2 inches long. The maximum volume is about 132 cubic inches. The dimensions of the box with this volume will be $x = 2$ inches by $10 - 2x = 6$ inches by $15 - 2x = 11$ inches.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain what a local maximum of a function is.

Concept Check ✓

2. Let f be a fourth-degree polynomial function with these zeros: 6, -2, $2i$, and $-2i$.
   a. How many distinct linear factors does $f(x)$ have?
   b. How many distinct solutions does $f(x) = 0$ have?
   c. What are the x-intercepts of the graph of f?
3. Let f be a fifth-degree polynomial function with five distinct real zeros. How many turning points does the graph of f have?

Skill Check ✓

Graph the function.

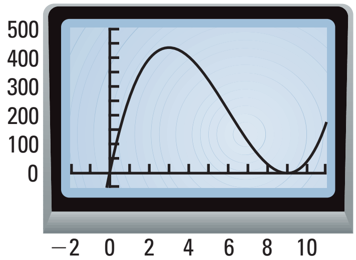
4. $f(x) = (x - 1)(x + 3)^2$
5. $f(x) = (x - 1)(x + 1)(x - 3)$
6. $f(x) = \frac{1}{8}(x + 1)(x - 1)(x - 3)$
7. $f(x) = \frac{1}{5}(x - 3)^2(x + 1)^2$

Use a graphing calculator to graph the function. Identify the *x*-intercepts and the points where the local maximums and local minimums occur.

8. $f(x) = 3x^4 - 5x^2 + 2x + 1$
9. $f(x) = x^3 - 3x^2 + x + 1$
10. $f(x) = -2x^3 + x^2 + 4x$
11. $f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 5x$
12. **MANUFACTURING** In Example 3, suppose you used a piece of cardboard that is 18 inches by 18 inches. Then the volume of the box would be given by this function:

$$V = 4x^3 - 72x^2 + 324x$$

Using a graphing calculator, you would obtain the graph shown at the right.



   a. What is the domain of the volume function? Explain.
   b. Use the graph to estimate the length of the cut that will maximize the volume of the box.
   c. Estimate the maximum volume the box can have.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 948.

GRAPHING POLYNOMIAL FUNCTIONS **Graph the function.**

13. $f(x) = (x - 1)^3(x + 1)$
14. $f(x) = \frac{1}{10}(x + 3)(x - 1)(x - 4)$
15. $f(x) = \frac{1}{8}(x + 4)(x + 2)(x - 3)$
16. $f(x) = 2(x + 2)^2(x + 4)^2$
17. $f(x) = 5(x - 1)(x - 2)(x - 3)$
18. $f(x) = \frac{1}{12}(x + 4)(x - 3)(x + 1)^2$
19. $f(x) = (x + 1)(x^2 - 3x + 3)$
20. $f(x) = (x + 2)(2x^2 - 2x + 1)$
21. $f(x) = (x - 2)(x^2 + x + 1)$
22. $f(x) = (x - 3)(x^2 - x + 1)$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 13–22
Example 2: Exs. 23–34
Example 3: Exs. 35–40

ANALYZING GRAPHS **Estimate the coordinates of each turning point and state whether each corresponds to a local maximum or a local minimum. Then list all the real zeros and determine the least degree that the function can have.**

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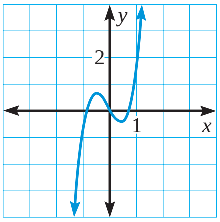

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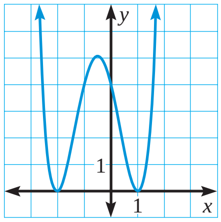

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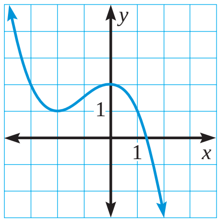

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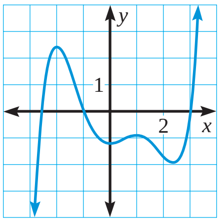

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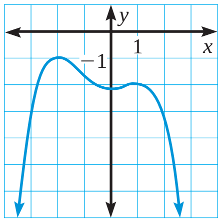

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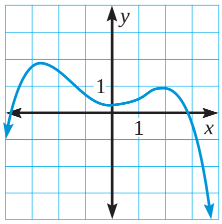


USING GRAPHS **Use a graphing calculator to graph the polynomial function. Identify the *x*-intercepts and the points where the local maximums and local minimums occur.**

29. $f(x) = 3x^3 - 9x + 1$

30. $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$

31. $f(x) = -\frac{1}{4}x^4 + 2x^2$

32. $f(x) = x^5 - 6x^3 + 9x$

33. $f(x) = x^5 - 5x^3 + 4x$

34. $f(x) = x^4 - 2x^3 - 3x^2 + 5x + 2$

35. **SWIMMING** The polynomial function

$$S = -241t^7 + 1062t^6 - 1871t^5 + 1647t^4 - 737t^3 + 144t^2 - 2.432t$$

models the speed S (in meters per second) of a swimmer doing the breast stroke during one complete stroke, where t is the number of seconds since the start of the stroke. Graph the function. At what time is the swimmer going the fastest?

36. **FOOD** The average amount of oranges (in pounds) eaten per person each year in the United States from 1991 to 1996 can be modeled by

$$f(x) = 0.298x^3 - 2.73x^2 + 7.05x + 8.45$$

where x is the number of years since 1991. Graph the function and identify any turning points on the interval $0 \le x \le 5$. What real-life meaning do these points have?

QUONSET HUTS **In Exercises 37–39, use the following information.**
A quonset hut is a dwelling shaped like half a cylinder. Suppose you have 600 square feet of material with which to build a quonset hut.

37. The formula for surface area is $S = \pi r^2 + \pi r l$ where r is the radius of the semicircle and l is the length of the hut. Substitute 600 for S and solve for l.

38. The formula for the volume of the hut is $V = \frac{1}{2}\pi r^2 l$. Write an equation for the volume V of the quonset hut as a polynomial function of r by substituting the expression for l from Exercise 37 into the volume formula.

39. Use the function you wrote in Exercise 38 to find the maximum volume of a quonset hut with a surface area of 600 square feet. What are the hut's dimensions?

FOCUS ON APPLICATIONS

QUONSET HUTS were invented during World War II. They were temporary structures that could be assembled quickly and easily. After the war they were sold as homes for about $1000 each.

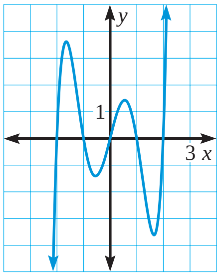
APPLICATION LINK
www.mcdougallittell.com

40. **CONSUMER ECONOMICS** The producer price index of butter from 1991 to 1997 can be modeled by $P = -0.233x^4 + 2.64x^3 - 6.59x^2 - 3.93x + 69.1$ where x is the number of years since 1991. Graph the function and identify any turning points on the interval $0 \le x \le 6$. What real-life meaning do these points have?

41. **CRITICAL THINKING** Sketch the graph of a polynomial function that has three turning points. Label each turning point as a local maximum or local minimum. What must be true about the degree of the polynomial function that has such a graph? Explain your reasoning.

Test Preparation

In Exercises 42 and 43, use the graph of the polynomial function *f* shown at the right.



42. **MULTIPLE CHOICE** What is the local maximum of f on the interval $-2 \le x \le -1$?

(A) $f(x) \approx 3.7$ (B) $f(x) \approx 1.4$
(C) $f(x) \approx -1.4$ (D) $f(x) \approx -3.7$

43. **MULTIPLE CHOICE** What is the local maximum of f on the interval $-1 \le x \le 1$?

(A) $f(x) \approx 3.7$ (B) $f(x) \approx 1.4$ (C) $f(x) \approx -1.4$ (D) $f(x) \approx -3.7$

★ **Challenge**

44. **GRAPHING OPPOSITES** Sketch the graph of $y = f(x)$ for this function:

$$f(x) = x^3 + 4x^2$$

Then sketch the graph of $y = -f(x)$. Explain how the graphs, the x-intercepts, the local maximums, and the local minimums are related. Finally, sketch the graph of $y = f(-x)$. Compare it with the others.

EXTRA CHALLENGE
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MIXED REVIEW

RELATING VARIABLES **The variables x and y vary directly. Write an equation that relates the variables.** (Review 2.4)

45. $x = 1, y = 7$
46. $x = -4, y = 6$
47. $x = 12, y = 3$
48. $x = 2, y = -5$
49. $x = -5, y = 3$
50. $x = -6, y = -15$

MATRIX PRODUCTS **Let A and B be matrices with the given dimensions. State whether the product AB is defined. If so, give the dimensions of AB.** (Review 4.2)

51. $A: 4 \times 3, B: 3 \times 1$
52. $A: 2 \times 4, B: 4 \times 5$
53. $A: 4 \times 3, B: 2 \times 4$
54. $A: 6 \times 6, B: 6 \times 5$

WRITING QUADRATIC FUNCTIONS **Write a quadratic function whose graph passes through the given points.** (Review 5.8 for 6.9)

55. vertex: $(1, 4)$; point: $(4, -5)$
56. vertex: $(-2, 6)$; point: $(0, 2)$
57. points: $(-5, 0), (5, 0), (7, 5)$
58. points: $(-2, 0), (4, 0), (1, -4)$

59. **PLANT GROWTH** You have a kudzu vine in your back yard. On Monday, the vine is 30 inches long. The following Thursday, the vine is 60 inches long. What is the average rate of change in the length of the vine? (Lesson 2.2)