6.7

Using the Fundamental Theorem of Algebra

What *you should learn*

GOAL 1 Use the fundamental theorem of algebra to determine the number of zeros of a polynomial function.

GOAL 2 Use technology to approximate the real zeros of a polynomial function, as applied in **Example 5**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding the American Indian, Aleut, and Eskimo population in **Ex. 59**.

REAL LIFE

GOAL 1 THE FUNDAMENTAL THEOREM OF ALGEBRA

The following important theorem, called the fundamental theorem of algebra, was first proved by the famous German mathematician Carl Friedrich Gauss (1777–1855).

THE FUNDAMENTAL THEOREM OF ALGEBRA

If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one root in the set of complex numbers.

In the following activity you will investigate how the number of solutions of $f(x) = 0$ is related to the degree of the polynomial $f(x)$.

ACTIVITY Developing Concepts

Investigating the Number of Solutions

1. Solve each polynomial equation. State how many solutions the equation has, and classify each as rational, irrational, or imaginary.

   a. $2x - 1 = 0$ **b.** $x^2 - 2 = 0$ **c.** $x^3 - 1 = 0$

   Make a conjecture about the relationship between the degree of a polynomial $f(x)$ and the number of solutions of $f(x) = 0$.

2. Solve the equation $x^3 + x^2 - x - 1 = 0$. How many different solutions are there? How can you reconcile this number with your conjecture?

The equation $x^3 - 6x^2 - 15x + 100 = 0$, which can be written as $(x + 4)(x - 5)^2 = 0$, has only two distinct solutions: -4 and 5. Because the factor $x - 5$ appears twice, however, you can count the solution 5 twice. So, with 5 counted as a **repeated solution**, this *third*-degree equation can be said to have *three* solutions: -4, 5, and 5.

In general, when all real and imaginary solutions are counted (with all repeated solutions counted individually), an nth-degree polynomial equation has *exactly* n solutions. Similarly, any nth-degree polynomial function has exactly n zeros.

EXAMPLE 1 ***Finding the Number of Solutions or Zeros***

a. The equation $x^3 + 3x^2 + 16x + 48 = 0$ has three solutions: -3, $4i$, and $-4i$.

b. The function $f(x) = x^4 + 6x^3 + 12x^2 + 8x$ has four zeros: -2, -2, -2, and 0.

EXAMPLE 2 *Finding the Zeros of a Polynomial Function*

Find all the zeros of $f(x) = x^5 - 2x^4 + 8x^2 - 13x + 6$.

SOLUTION

The possible rational zeros are ±1, ±2, ±3, and ±6. Using synthetic division, you can determine that 1 is a repeated zero and that −2 is also a zero. You can write the function in factored form as follows:

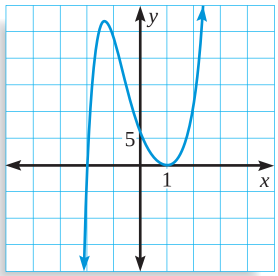
$$f(x) = (x - 1)(x - 1)(x + 2)(x^2 - 2x + 3)$$

Complete the factorization, using the quadratic formula to factor the trinomial.

$$f(x) = (x - 1)(x - 1)(x + 2)[x - (1 + i\sqrt{2})][x - (1 - i\sqrt{2})]$$

▶ This factorization gives the following five zeros:

$$1,\ 1,\ -2,\ 1 + i\sqrt{2},\text{ and } 1 - i\sqrt{2}$$



The graph of f is shown at the right. Note that only the *real* zeros appear as x-intercepts. Also note that the graph only *touches* the x-axis at the repeated zero $x = 1$, but *crosses* the x-axis at the zero $x = -2$.

....

The graph in Example 2 illustrates the behavior of the graph of a polynomial function near its zeros. When a factor $x - k$ is raised to an odd power, the graph crosses the x-axis at $x = k$. When a factor $x - k$ is raised to an even power, the graph is tangent to the x-axis at $x = k$.

In Example 2 the zeros $\mathbf{1 + i\sqrt{2}}$ and $\mathbf{1 - i\sqrt{2}}$ are complex conjugates. The complex zeros of a polynomial function with *real* coefficients always occur in complex conjugate pairs. That is, if $a + bi$ is a zero, then $a - bi$ must also be a zero.

EXAMPLE 3 *Using Zeros to Write Polynomial Functions*

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

Write a polynomial function f of least degree that has real coefficients, a leading coefficient of 1, and 2 and $1 + i$ as zeros.

SOLUTION

Because the coefficients are real and $1 + i$ is a zero, $1 - i$ must also be a zero. Use the three zeros and the factor theorem to write $f(x)$ as a product of three factors.

$f(x) = (x - 2)[x - (1 + i)][x - (1 - i)]$	**Write $f(x)$ in factored form.**
$= (x - 2)[(x - 1) - i][(x - 1) + i]$	**Regroup terms.**
$= (x - 2)[(x - 1)^2 - i^2]$	**Multiply.**
$= (x - 2)[x^2 - 2x + 1 - (-1)]$	**Expand power and use $i^2 = -1$.**
$= (x - 2)(x^2 - 2x + 2)$	**Simplify.**
$= x^3 - 2x^2 + 2x - 2x^2 + 4x - 4$	**Multiply.**
$= x^3 - 4x^2 + 6x - 4$	**Combine like terms.**

✓**CHECK** You can check this result by evaluating $f(x)$ at each of its three zeros.

GOAL 2 Using Technology to Approximate Zeros

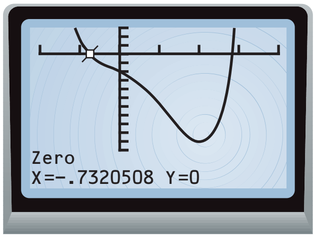
The rational zero theorem gives you a way to find the rational zeros of a polynomial function with integer coefficients. To find the *real* zeros of *any* polynomial function, you may need to use technology.

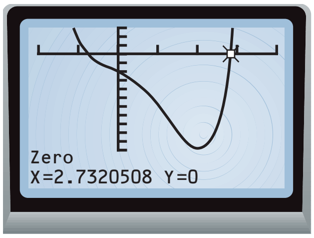
EXAMPLE 4 *Approximating Real Zeros*

Approximate the real zeros of $f(x) = x^4 - 2x^3 - x^2 - 2x - 2$.

Solution

There are several ways to use a graphing calculator to approximate the real zeros of a function. One way is to use the *Zero* (or *Root*) feature as shown below.





▶ From these screens, you can see that the real zeros are about -0.73 and 2.73.

Because the polynomial function has degree 4, you know that there must be two other zeros. These may be repeats of the real zeros, or they may be imaginary zeros. In this particular case, the two other zeros are imaginary: $x = \pm i$.

Focus on Applications

Real Life **HARVARD STEP TEST** When taking the Harvard Step Test, a person steps up and down a 20 inch platform for 5 minutes. The person's score is determined by his or her heart rate in the first few minutes after stopping.

EXAMPLE 5 *Approximating Real Zeros of a Real-Life Function*

Physiology For one group of people it was found that a person's score S on the Harvard Step Test was related to his or her amount of hemoglobin x (in grams per 100 milliliters of blood) by the following model:

$$S = -0.015x^3 + 0.6x^2 - 2.4x + 19$$

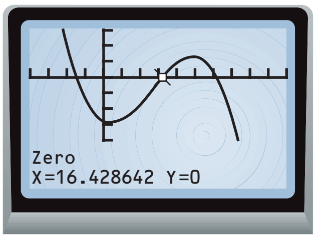
The normal range of hemoglobin is 12–18 grams per 100 milliliters of blood. Approximate the amount of hemoglobin for a person who scored 75.

Solution

You can solve the equation

$$75 = -0.015x^3 + 0.6x^2 - 2.4x + 19$$

by rewriting it as $0 = -0.015x^3 + 0.6x^2 - 2.4x - 56$ and then using a graphing calculator to approximate the real zeros of $f(x) = -0.015x^3 + 0.6x^2 - 2.4x - 56$. From the graph you can see that there are three real zeros: $x \approx -7.3$, $x \approx 16.4$, and $x \approx 30.9$.



▶ The person's hemoglobin is probably about 16.4 grams per 100 milliliters of blood, since this is the only zero within the normal range.

GUIDED PRACTICE

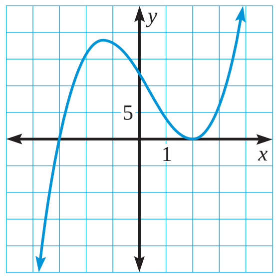
Vocabulary Check ✓

Concept Check ✓

1. State the fundamental theorem of algebra.

2. Two zeros of $f(x) = x^3 - 6x^2 - 16x + 96$ are 4 and -4. Explain why the third zero must also be a real number.

3. The graph of $f(x) = x^3 - x^2 - 8x + 12$ is shown at the right. How many real zeros does the function have? How many imaginary zeros does the function have? Explain your reasoning.



Ex. 3

Skill Check ✓

Find all the zeros of the polynomial function.

4. $f(x) = x^3 - x^2 - 2x$

5. $f(x) = x^4 + x^2 - 12$

6. $f(x) = x^3 + 5x^2 - 9x - 45$

7. $f(x) = x^4 - x^3 + 2x^2 - 4x - 8$

Write a polynomial function of least degree that has real coefficients, the given zeros, and a leading coefficient of 1.

8. 3, 0, -2

9. 1, 1, i, $-i$

10. 5, $2 + 3i$

11. 1, -1, 2, -2, 3

12. 3, -2, $-1 + i$

13. $4i$, $4i$

14. **GROCERY STORE REVENUE** For the 25 years that a grocery store has been open, its annual revenue R (in millions of dollars) can be modeled by

$$R = \frac{1}{10{,}000}(-t^4 + 12t^3 - 77t^2 + 600t + 13{,}650)$$

where t is the number of years the store has been open. In what year(s) was the revenue \$1.5 million?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 948.

CHECKING ZEROS **Decide whether the given *x*-value is a zero of the function.**

15. $f(x) = x^3 - x^2 + 4x - 4,\ x = 1$

16. $f(x) = x^3 + 3x^2 - 5x + 8,\ x = 4$

17. $f(x) = x^4 - x^2 - 3x + 3,\ x = 0$

18. $f(x) = x^3 + 5x^2 + x + 5,\ x = -5$

19. $f(x) = x^3 - 4x^2 + 16x - 64,\ x = 4i$

20. $f(x) = x^3 - 3x^2 + x - 3,\ x = -i$

FINDING ZEROS **Find all the zeros of the polynomial function.**

21. $f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$

22. $f(x) = x^4 + 4x^3 - 6x^2 - 36x - 27$

23. $f(x) = x^3 - 4x^2 + 3x$

24. $f(x) = x^3 + 5x^2 - 4x - 20$

25. $f(x) = x^4 + 7x^3 - x^2 - 67x - 60$

26. $f(x) = x^4 - 5x^2 - 36$

27. $f(x) = x^3 - x^2 + 49x - 49$

28. $f(x) = x^3 - x^2 + 25x - 25$

29. $f(x) = x^4 + 6x^3 + 14x^2 + 54x + 45$

30. $f(x) = x^3 + 3x^2 + 25x + 75$

31. $f(x) = x^4 - x^3 - 5x^2 - x - 6$

32. $f(x) = x^4 + x^3 + 2x^2 + 4x - 8$

33. $f(x) = 2x^4 - 7x^3 - 27x^2 + 63x + 81$

34. $f(x) = 2x^4 - x^3 - 42x^2 + 16x + 160$

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 21–54
Example 2: Exs. 21–34
Example 3: Exs. 35–46
Example 4: Exs. 47–54
Example 5: Exs. 55–59

FOCUS ON APPLICATIONS

REAL LIFE **UNITED STATES EXPORTS** The United States exports more than any other country in the world. It also imports more than any other country.

WRITING POLYNOMIAL FUNCTIONS **Write a polynomial function of least degree that has real coefficients, the given zeros, and a leading coefficient of 1.**

35. 2, 1, 4 **36.** 1, -4, 5 **37.** -6, 3, 5

38. -5, 2, -2 **39.** -2, -4, -7 **40.** 8, $-i$, i

41. $3i$, $-3i$, 5 **42.** 2, -2, $-6i$ **43.** i, $-3i$, $3i$

44. $3 - i$, $5i$ **45.** 4, 4, $2 + i$ **46.** -2, -2, 3, $-4i$

FINDING ZEROS **Use a graphing calculator to graph the polynomial function. Then use the *Zero* (or *Root*) feature of the calculator to find the real zeros of the function.**

47. $f(x) = x^3 - x^2 - 5x + 3$ **48.** $f(x) = 2x^3 - x^2 - 3x - 1$

49. $f(x) = x^3 - 2x^2 + x + 1$ **50.** $f(x) = x^4 - 2x - 1$

51. $f(x) = x^4 - x^3 - 4x^2 - 3x - 2$ **52.** $f(x) = x^4 - x^3 - 3x^2 - x + 1$

53. $f(x) = x^4 + 3x^2 - 2$ **54.** $f(x) = x^4 - x^3 - 20x^2 + 10x + 27$

GRAPHING MODELS **In Exercises 55–59, you may find it helpful to graph the model on a graphing calculator.**

55. **UNITED STATES EXPORTS** For 1980 through 1996, the total exports E (in billions of dollars) of the United States can be modeled by

$$E = -0.131t^3 + 5.033t^2 - 23.2t + 233$$

where t is the number of years since 1980. In what year were the total exports about \$312.76 billion? ▶ Source: U.S. Bureau of the Census

56. **EDUCATION DONATIONS** For 1983 through 1995, the amount of private donations D (in millions of dollars) allocated to education can be modeled by

$$D = 1.78t^3 - 6.02t^2 + 752t + 6701$$

where t is the number of years since 1983. In what year was \$14.3 billion of private donations allocated to education? ▶ Source: AAFRC Trust for Philanthropy

57. **SPORTS EQUIPMENT** For 1987 through 1996, the sales S (in millions of dollars) of gym shoes and sneakers can be modeled by

$$S = -0.982t^5 + 24.6t^4 - 211t^3 + 661t^2 - 318t + 1520$$

where t is the number of years since 1987. Were there any years in which sales were about \$2 billion? Explain. ▶ Source: National Sporting Goods Association

58. **TELEVISION** For 1990 through 2000, the actual and projected amount spent on television per person per year in the United States can be modeled by

$$S = -0.213t^3 + 3.96t^2 + 10.2t + 366$$

where S is the amount spent (in dollars) and t is the number of years since 1990. During which year was \$455 spent per person on television?
▶ Source: Veronis, Suhler & Associates, Inc.

59. **POPULATION** For 1890 through 1990, the American Indian, Eskimo, and Aleut population P (in thousands) can be modeled by the function

$$P = 0.00496t^3 - 0.432t^2 + 11.3t + 212$$

where t is the number of years since 1890. In what year did the population reach 722,000? INTERNET **DATA UPDATE** of *Statistical Abstract of the United States* data at www.mcdougallittell.com

Test Preparation

60. MULTI-STEP PROBLEM Mary plans to save \$1000 each summer to buy a used car at the end of the fourth summer. At the end of each summer, she will deposit the \$1000 she earned from her summer job into her bank account. The table shows the value of her deposits over the four year period. In the table, g is the growth factor $1 + r$ where r is the annual interest rate expressed as a decimal.

	End of 1st summer	End of 2nd summer	End of 3rd summer	End of 4th summer
Value of 1st deposit	1000	$1000g$	$1000g^2$	$1000g^3$
Value of 2nd deposit	—	1000	?	?
Value of 3rd deposit	—	—	1000	?
Value of 4th deposit	—	—	—	1000

a. Copy and complete the table.

b. Write a polynomial function of g that represents the value of Mary's account at the end of the fourth summer.

c. *Writing* Suppose Mary wants to buy a car that costs about \$4300. What growth factor does she need to obtain this amount? What annual interest rate does she need? Explain how you found your answers.

★ Challenge

61. a. Copy and complete the table.

Function	Zeros	Sum of zeros	Product of zeros
$f(x) = x^2 - 5x + 6$	?	?	?
$f(x) = x^3 - 7x + 6$	?	?	?
$f(x) = x^4 + 2x^3 + x^2 + 8x - 12$	?	?	?
$f(x) = x^5 - 3x^4 - 9x^3 + 25x^2 - 6x$	?	?	?

b. Use your completed table to make a conjecture relating the sum of the zeros of a polynomial function with the coefficients of the polynomial function.

c. Use your completed table to make a conjecture relating the product of the zeros of a polynomial function with the coefficients of the polynomial function.

EXTRA CHALLENGE
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62. Show that the sum of a pair of complex conjugates is a real number.

63. Show that the product of a pair of complex conjugates is a real number.

MIXED REVIEW

GRAPHING WITH INTERCEPT FORM **Graph the quadratic function. Label the vertex, axis of symmetry, and x-intercepts.** (Review 5.1 for 6.8)

64. $y = -3(x - 2)(x + 2)$

65. $y = 2(x - 1)(x - 5)$

66. $y = 2(x + 4)(x - 3)$

67. $y = -(x + 1)(x - 5)$

GRAPHING POLYNOMIALS **Graph the polynomial function.** (Review 6.2 for 6.8)

68. $f(x) = -2x^4$

69. $f(x) = -x^3 - 4$

70. $f(x) = x^3 + 4x - 3$

71. $f(x) = x^4 - 3x^3 + x + 2$