

# 5.7

## Graphing and Solving Quadratic Inequalities

### What you should learn

**GOAL 1** Graph quadratic inequalities in two variables.

**GOAL 2** Solve quadratic inequalities in one variable, as applied in **Example 7**.

### Why you should learn it

▼ To solve **real-life** problems, such as finding the weight of theater equipment that a rope can support in **Exs. 47 and 48**.



### GOAL 1 QUADRATIC INEQUALITIES IN TWO VARIABLES

In this lesson you will study four types of **quadratic inequalities in two variables**.

$$y < ax^2 + bx + c$$

$$y \leq ax^2 + bx + c$$

$$y > ax^2 + bx + c$$

$$y \geq ax^2 + bx + c$$

The graph of any such inequality consists of all solutions  $(x, y)$  of the inequality. The steps used to graph a quadratic inequality are very much like those used to graph a linear inequality. (See Lesson 2.6.)

#### GRAPHING A QUADRATIC INEQUALITY IN TWO VARIABLES

To graph one of the four types of quadratic inequalities shown above, follow these steps:

- STEP 1** Draw the parabola with equation  $y = ax^2 + bx + c$ . Make the parabola *dashed* for inequalities with  $<$  or  $>$  and *solid* for inequalities with  $\leq$  or  $\geq$ .
- STEP 2** Choose a point  $(x, y)$  inside the parabola and check whether the point is a solution of the inequality.
- STEP 3** If the point from Step 2 is a solution, shade the region inside the parabola. If it is not a solution, shade the region outside the parabola.

#### EXAMPLE 1 Graphing a Quadratic Inequality

Graph  $y > x^2 - 2x - 3$ .

#### SOLUTION

Follow Steps 1–3 listed above.

- 1** Graph  $y = x^2 - 2x - 3$ . Since the inequality symbol is  $>$ , make the parabola dashed.
- 2** Test a point inside the parabola, such as  $(1, 0)$ .

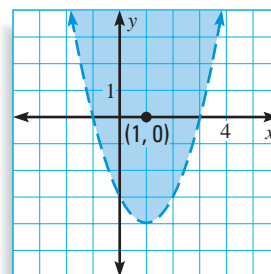
$$y > x^2 - 2x - 3$$

$$0 \stackrel{?}{>} 1^2 - 2(1) - 3$$

$$0 > -4 \checkmark$$

So,  $(1, 0)$  is a solution of the inequality.

- 3** Shade the region inside the parabola.



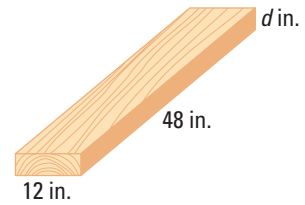


### EXAMPLE 2 Using a Quadratic Inequality as a Model

You are building a wooden bookcase. You want to choose a thickness  $d$  (in inches) for the shelves so that each is strong enough to support 60 pounds of books without breaking. A shelf can safely support a weight of  $W$  (in pounds) provided that:

$$W \leq 300d^2$$

- Graph the given inequality.
- If you make each shelf 0.75 inch thick, can it support a weight of 60 pounds?



#### SOLUTION

- Graph  $W = 300d^2$  for nonnegative values of  $d$ . Since the inequality symbol is  $\leq$ , make the parabola solid. Test a point inside the parabola, such as  $(0.5, 240)$ .

$$W \leq 300d^2$$

$$240 \stackrel{?}{\leq} 300(0.5)^2$$

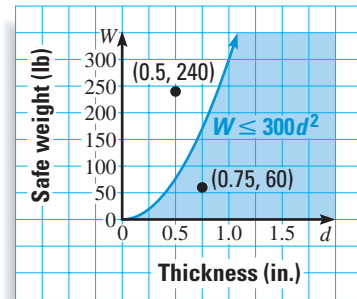
$$240 \neq 75$$

Since the chosen point is not a solution, shade the region outside (below) the parabola.

- The point  $(0.75, 60)$  lies in the shaded region of the graph from part (a), so  $(0.75, 60)$  is a solution of the given inequality. Therefore, a shelf that is 0.75 inch thick *can* support a weight of 60 pounds.

.....

Graphing a *system* of quadratic inequalities is similar to graphing a system of linear inequalities. First graph each inequality in the system. Then identify the region in the coordinate plane common to all the graphs. This region is called the *graph of the system*.



### EXAMPLE 3 Graphing a System of Quadratic Inequalities

Graph the system of quadratic inequalities.

$$y \geq x^2 - 4 \quad \text{Inequality 1}$$

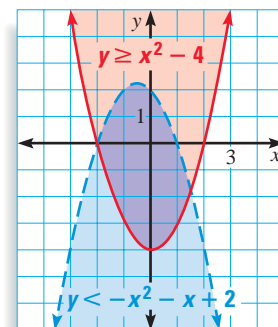
$$y < -x^2 - x + 2 \quad \text{Inequality 2}$$

#### SOLUTION

**Graph** the inequality  $y \geq x^2 - 4$ . The graph is the red region inside and including the parabola  $y = x^2 - 4$ .

**Graph** the inequality  $y < -x^2 - x + 2$ . The graph is the blue region inside (but not including) the parabola  $y = -x^2 - x + 2$ .

**Identify** the **purple region** where the two graphs overlap. This region is the graph of the system.



#### STUDENT HELP

##### Look Back

For help with graphing inequalities in two variables, see p. 108.

## GOAL 2 QUADRATIC INEQUALITIES IN ONE VARIABLE

One way to solve a **quadratic inequality in one variable** is to use a graph.

- To solve  $ax^2 + bx + c < 0$  (or  $ax^2 + bx + c \leq 0$ ), graph  $y = ax^2 + bx + c$  and identify the  $x$ -values for which the graph lies *below* (or *on and below*) the  $x$ -axis.
- To solve  $ax^2 + bx + c > 0$  (or  $ax^2 + bx + c \geq 0$ ), graph  $y = ax^2 + bx + c$  and identify the  $x$ -values for which the graph lies *above* (or *on and above*) the  $x$ -axis.

### EXAMPLE 4 Solving a Quadratic Inequality by Graphing

#### STUDENT HELP

#### Look Back

For help with solving inequalities in one variable, see p. 41.

Solve  $x^2 - 6x + 5 < 0$ .

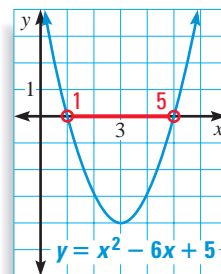
#### SOLUTION

The solution consists of the  $x$ -values for which the graph of  $y = x^2 - 6x + 5$  lies below the  $x$ -axis. Find the graph's  $x$ -intercepts by letting  $y = 0$  and using factoring to solve for  $x$ .

$$0 = x^2 - 6x + 5$$

$$0 = (x - 1)(x - 5)$$

$$x = 1 \text{ or } x = 5$$



Sketch a parabola that opens up and has 1 and 5 as  $x$ -intercepts. The graph lies below the  $x$ -axis between  $x = 1$  and  $x = 5$ .

► The solution of the given inequality is  $1 < x < 5$ .

### EXAMPLE 5 Solving a Quadratic Inequality by Graphing

Solve  $2x^2 + 3x - 3 \geq 0$ .

#### SOLUTION

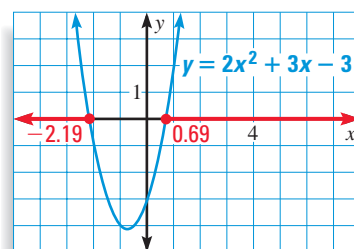
The solution consists of the  $x$ -values for which the graph of  $y = 2x^2 + 3x - 3$  lies on and above the  $x$ -axis. Find the graph's  $x$ -intercepts by letting  $y = 0$  and using the quadratic formula to solve for  $x$ .

$$0 = 2x^2 + 3x - 3$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{33}}{4}$$

$$x \approx 0.69 \text{ or } x \approx -2.19$$



Sketch a parabola that opens up and has 0.69 and  $-2.19$  as  $x$ -intercepts. The graph lies on and above the  $x$ -axis to the left of (and including)  $x = -2.19$  and to the right of (and including)  $x = 0.69$ .

► The solution of the given inequality is approximately  $x \leq -2.19$  or  $x \geq 0.69$ .

You can also use an algebraic approach to solve a quadratic inequality in one variable, as demonstrated in Example 6.

### EXAMPLE 6 Solving a Quadratic Inequality Algebraically

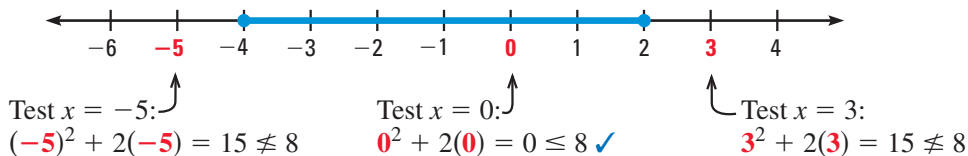
Solve  $x^2 + 2x \leq 8$ .

#### SOLUTION

First write and solve the equation obtained by replacing the inequality symbol with an equals sign.

$$\begin{array}{ll} x^2 + 2x \leq 8 & \text{Write original inequality.} \\ x^2 + 2x = 8 & \text{Write corresponding equation.} \\ x^2 + 2x - 8 = 0 & \text{Write in standard form.} \\ (x + 4)(x - 2) = 0 & \text{Factor.} \\ x = -4 \text{ or } x = 2 & \text{Zero product property} \end{array}$$

The numbers  $-4$  and  $2$  are called the *critical  $x$ -values* of the inequality  $x^2 + 2x \leq 8$ . Plot  $-4$  and  $2$  on a number line, using solid dots because the values satisfy the inequality. The critical  $x$ -values partition the number line into three intervals. Test an  $x$ -value in each interval to see if it satisfies the inequality.



▶ The solution is  $-4 \leq x \leq 2$ .

### EXAMPLE 7 Using a Quadratic Inequality as a Model

**DRIVING** For a driver aged  $x$  years, a study found that the driver's reaction time  $V(x)$  (in milliseconds) to a visual stimulus such as a traffic light can be modeled by:

$$V(x) = 0.005x^2 - 0.23x + 22, \quad 16 \leq x \leq 70$$

At what ages does a driver's reaction time tend to be greater than 25 milliseconds?

▶ Source: *Science Probe!*

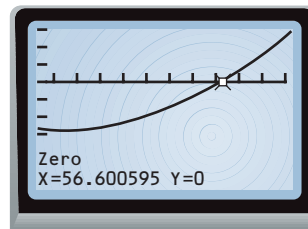
#### SOLUTION

You want to find the values of  $x$  for which:

$$\begin{aligned} V(x) &> 25 \\ 0.005x^2 - 0.23x + 22 &> 25 \\ 0.005x^2 - 0.23x - 3 &> 0 \end{aligned}$$

Graph  $y = 0.005x^2 - 0.23x - 3$  on the domain  $16 \leq x \leq 70$ . The graph's  $x$ -intercept is about 57, and the graph lies above the  $x$ -axis when  $57 < x \leq 70$ .

▶ Drivers over 57 years old tend to have reaction times greater than 25 milliseconds.



#### FOCUS ON APPLICATIONS



**REAL LIFE DRIVING** Driving simulators help drivers safely improve their reaction times to hazardous situations they may encounter on the road.

# GUIDED PRACTICE

## Vocabulary Check ✓

1. Give one example each of a quadratic inequality in one variable and a quadratic inequality in two variables.

## Concept Check ✓

2. How does the graph of  $y > x^2$  differ from the graph of  $y \geq x^2$ ?

3. Explain how to solve  $x^2 - 3x - 4 > 0$  graphically and algebraically.

## Skill Check ✓

Graph the inequality.

4.  $y \geq x^2 + 2$

5.  $y \leq -2x^2$

6.  $y < x^2 - 5x + 4$

Graph the system of inequalities.

7.  $y \leq -x^2 + 3$   
 $y \geq x^2 + 2x - 4$

8.  $y \geq -x^2 + 3$   
 $y \geq x^2 + 2x - 4$

9.  $y \geq -x^2 + 3$   
 $y \leq x^2 + 2x - 4$

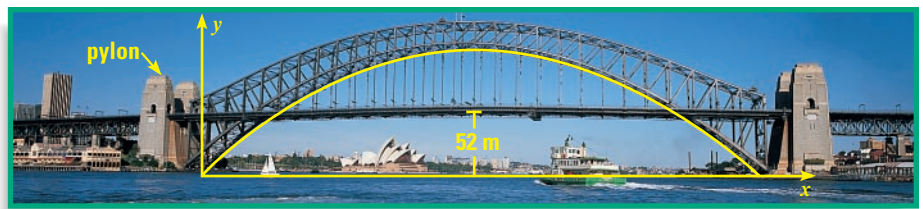
Solve the inequality.

10.  $x^2 - 4 < 0$

11.  $x^2 - 4 \geq 0$

12.  $x^2 - 4 > 3x$

13. **ARCHITECTURE** The arch of the Sydney Harbor Bridge in Sydney, Australia, can be modeled by  $y = -0.00211x^2 + 1.06x$  where  $x$  is the distance (in meters) from the left pylons and  $y$  is the height (in meters) of the arch above the water. For what distances  $x$  is the arch above the road?



# PRACTICE AND APPLICATIONS

## STUDENT HELP

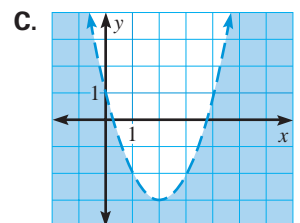
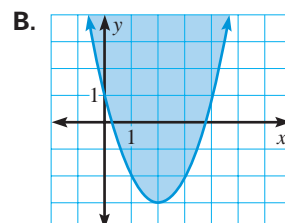
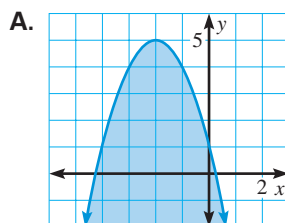
➔ **Extra Practice** to help you master skills is on p. 947.

**MATCHING GRAPHS** Match the inequality with its graph.

14.  $y \geq x^2 - 4x + 1$

15.  $y < x^2 - 4x + 1$

16.  $y \leq -x^2 - 4x + 1$



## STUDENT HELP

### ➔ HOMEWORK HELP

**Example 1:** Exs. 14–28

**Example 2:** Exs. 47–49

**Example 3:** Exs. 29–34, 49

**Examples 4, 5:** Exs. 35–40

**Example 6:** Exs. 41–46

**Example 7:** Exs. 50, 51

**GRAPHING QUADRATIC INEQUALITIES** Graph the inequality.

17.  $y \geq 3x^2$

18.  $y \leq -x^2$

19.  $y > -x^2 + 5$

20.  $y < x^2 - 3x$

21.  $y \leq x^2 + 8x + 16$

22.  $y \leq -x^2 + x + 6$

23.  $y \geq 2x^2 - 2x - 5$

24.  $y \geq -2x^2 - x + 3$

25.  $y > -3x^2 + 5x - 4$

26.  $y < -\frac{1}{2}x^2 - 2x + 4$

27.  $y > \frac{4}{3}x^2 - 12x + 29$

28.  $y < 0.6x^2 + 3x + 2.4$

**FOCUS ON CAREERS**



**SET DESIGNER**

A set designer creates the scenery, or *sets*, used in a theater production. The designer may make scale models of the sets before they are actually built.



**CAREER LINK**

[www.mcdougallittell.com](http://www.mcdougallittell.com)

**GRAPHING SYSTEMS** Graph the system of inequalities.

- |  |   |   |
|--|---|---|
| 29. $y \geq x^2$<br>$y \leq x^2 + 3$               | 30. $y < -3x^2$<br>$y \geq -\frac{1}{2}x^2 - 5$ | 31. $y > x^2 - 6x + 9$<br>$y < -x^2 + 6x - 3$     |
| 32. $y \geq x^2 + 2x + 1$<br>$y \geq x^2 - 4x + 4$ | 33. $y < 3x^2 + 2x - 5$<br>$y \geq -2x^2 + 1$   | 34. $y \leq 2x^2 - 9x + 8$<br>$y > -x^2 - 6x - 4$ |

**SOLVING BY GRAPHING** Solve the inequality by graphing.

- |                        |                            |                                    |
|------------------------|----------------------------|------------------------------------|
| 35. $x^2 + x - 2 < 0$  | 36. $2x^2 - 7x + 3 \geq 0$ | 37. $-x^2 - 2x + 8 \leq 0$         |
| 38. $-x^2 + x + 5 > 0$ | 39. $3x^2 + 24x \geq -41$  | 40. $-\frac{3}{4}x^2 + 4x - 8 < 0$ |

**SOLVING ALGEBRAICALLY** Solve the inequality algebraically.

- |                            |                             |                                   |
|----------------------------|-----------------------------|-----------------------------------|
| 41. $x^2 + 3x - 18 \geq 0$ | 42. $3x^2 - 16x + 5 \leq 0$ | 43. $4x^2 < 25$                   |
| 44. $-x^2 - 12x < 32$      | 45. $2x^2 - 4x - 5 > 0$     | 46. $\frac{1}{2}x^2 + 3x \leq -6$ |

**THEATER** In Exercises 47 and 48, use the following information.

You are a member of a theater production crew. You use manila rope and wire rope to support lighting, scaffolding, and other equipment. The weight  $W$  (in pounds) that can be safely supported by a rope with diameter  $d$  (in inches) is given below for both types of rope. ▶ Source: *Workshop Math*

**Manila rope:**  $W \leq 1480d^2$       **Wire rope:**  $W \leq 8000d^2$

47. Graph the inequalities in separate coordinate planes for  $0 \leq d \leq 1\frac{1}{2}$ .
48. Based on your graphs, can 1000 pounds of theater equipment be supported by a  $\frac{1}{2}$  inch manila rope? by a  $\frac{1}{2}$  inch wire rope?
49. **HEALTH** For a person of height  $h$  (in inches), a healthy weight  $W$  (in pounds) is one that satisfies this system of inequalities:

$$W \geq \frac{19h^2}{703} \quad \text{and} \quad W \leq \frac{25h^2}{703}$$

Graph the system for  $0 \leq h \leq 80$ . What is the range of healthy weights for a person 67 inches tall? ▶ Source: *Parade Magazine*

**SOLVING INEQUALITIES** In Exercises 50–52, you may want to use a graphing calculator to help you solve the problems.

50. **FORESTRY** *Sawtimber* is a term for trees that are suitable for sawing into lumber, plywood, and other products. For the years 1983–1995, the unit value  $y$  (in 1994 dollars per million board feet) of one type of sawtimber harvested in California can be modeled by

$$y = 0.125x^2 - 569x + 848,000, \quad 400 \leq x \leq 2200$$

where  $x$  is the volume of timber harvested (in millions of board feet).

▶ Source: California Department of Forestry and Fire Protection

- a. For what harvested timber volumes is the value of the timber at least \$400,000 per million board feet?
- b. **LOGICAL REASONING** What happens to the unit value of the timber as the volume harvested increases? Why would you expect this to happen?

**STUDENT HELP**

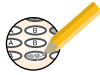


**HOMEWORK HELP**

Visit our Web site [www.mcdougallittell.com](http://www.mcdougallittell.com) for help with problem solving in Exs. 50–52.



## Test Preparation



51. **MEDICINE** In 1992 the average income  $I$  (in dollars) for a doctor aged  $x$  years could be modeled by:

$$I = -425x^2 + 42,500x - 761,000$$

For what ages did the average income for a doctor exceed \$250,000?



**DATA UPDATE** of American Almanac of Jobs and Salaries data at [www.mcdougallittell.com](http://www.mcdougallittell.com)

52. **MULTI-STEP PROBLEM** A study of driver reaction times to audio stimuli found that the reaction time  $A(x)$  (in milliseconds) of a driver can be modeled by

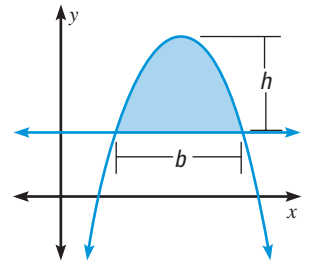
$$A(x) = 0.0051x^2 - 0.319x + 15, \quad 16 \leq x \leq 70$$

where  $x$  is the driver's age (in years). **Source:** *Science Probe!*

- Graph  $y = A(x)$  on the given domain. Also graph  $y = V(x)$ , the reaction-time model for visual stimuli from Example 7, in the same coordinate plane.
- For what values of  $x$  in the interval  $16 \leq x \leq 70$  is  $A(x) < V(x)$ ?
- Writing** Based on your results from part (b), do you think a driver would react more quickly to a traffic light changing from green to yellow or to the siren of an approaching ambulance? Explain.

## ★ Challenge

53. **GEOMETRY CONNECTION** The area  $A$  of the region bounded by a parabola and a horizontal line is given by  $A = \frac{2}{3}bh$  where  $b$  and  $h$  are as defined in the diagram. Find the area of the region determined by each pair of inequalities.



- $y \leq -x^2 + 4x$   
 $y \geq 0$
- $y \geq x^2 - 4x - 5$   
 $y \leq 3$

### EXTRA CHALLENGE

[www.mcdougallittell.com](http://www.mcdougallittell.com)

## MIXED REVIEW

**SOLVING FOR A VARIABLE** Solve the equation for  $y$ . (Review 1.4)

- $3x + y = 1$
- $8x - 2y = 10$
- $-2x + 5y = 9$
- $\frac{1}{6}x + \frac{1}{3}y = -\frac{11}{12}$
- $xy - x = 2$
- $\frac{x - 3y}{4} = 7x$

**SOLVING SYSTEMS** Solve the system of linear equations. (Review 3.6 for 5.8)

- $5x - 3y - 2z = -17$   
 $-x + 7y - 3z = 6$   
 $3x + 2y + 4z = 13$
- $x - 4y + z = -14$   
 $2x + 3y + 7z = -15$   
 $-3x + 5y - 5z = 29$

**COMPLEX NUMBERS** Write the expression as a complex number in standard form. (Review 5.4)

- $(3 + 4i) + (10 - i)$
- $(-11 - 2i) + (5 + 2i)$
- $(9 + i) - (4 - i)$
- $(5 - 3i) - (-1 + 2i)$
- $6i(8 + i)$
- $(7 + 3i)(2 - 5i)$
- $\frac{1}{3 - i}$
- $\frac{4 - 3i}{9 + 2i}$