5.3 Solving Quadratic Equations by Finding Square Roots

What you should learn

GOAL 1 Solve quadratic equations by finding square roots.

GOAL 2 Use quadratic equations to solve **real-life** problems, such as finding how long a falling stunt man is in the air in **Example 4**.

Why you should learn it

▼ To model **real-life** quantities, such as the height of a rock dropped off the Leaning Tower of Pisa in **Ex. 69**.

REAL LIFE

GOAL 1 SOLVING QUADRATIC EQUATIONS

A number r is a **square root** of a number s if $r^2 = s$. A positive number s has two square roots denoted by $\sqrt{s}$ and $-\sqrt{s}$. The symbol $\sqrt{}$ is a **radical sign**, the number s beneath the radical sign is the **radicand**, and the expression $\sqrt{s}$ is a **radical**.

For example, since $3^2 = 9$ and $(-3)^2 = 9$, the two square roots of 9 are $\sqrt{9} = 3$ and $-\sqrt{9} = -3$. You can use a calculator to approximate $\sqrt{s}$ when s is not a perfect square. For instance, $\sqrt{2} \approx 1.414$.

ACTIVITY — Developing Concepts: Investigating Properties of Square Roots

1. Evaluate the two expressions. What do you notice about the square root of a product of two numbers?

   a. $\sqrt{36}$, $\sqrt{4} \cdot \sqrt{9}$ **b.** $\sqrt{8}$, $\sqrt{4} \cdot \sqrt{2}$ **c.** $\sqrt{30}$, $\sqrt{3} \cdot \sqrt{10}$

2. Evaluate the two expressions. What do you notice about the square root of a quotient of two numbers?

   a. $\sqrt{\frac{4}{9}}$, $\frac{\sqrt{4}}{\sqrt{9}}$ **b.** $\sqrt{\frac{25}{2}}$, $\frac{\sqrt{25}}{\sqrt{2}}$ **c.** $\sqrt{\frac{19}{7}}$, $\frac{\sqrt{19}}{\sqrt{7}}$

In the activity you may have discovered the following properties of square roots. You can use these properties to simplify expressions containing square roots.

PROPERTIES OF SQUARE ROOTS ($a > 0$, $b > 0$)

Product Property: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

Quotient Property: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

A square-root expression is considered simplified if (1) no radicand has a perfect-square factor other than 1, and (2) there is no radical in a denominator.

EXAMPLE 1 *Using Properties of Square Roots*

Simplify the expression.

a. $\sqrt{24} = \sqrt{4} \cdot \sqrt{6} = 2\sqrt{6}$

b. $\sqrt{6} \cdot \sqrt{15} = \sqrt{90} = \sqrt{9} \cdot \sqrt{10} = 3\sqrt{10}$

c. $\sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{\sqrt{16}} = \frac{\sqrt{7}}{4}$

d. $\sqrt{\frac{7}{2}} = \frac{\sqrt{7}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{14}}{2}$

In part (d) of Example 1, the square root in the denominator of $\frac{\sqrt{7}}{\sqrt{2}}$ was eliminated by multiplying both the numerator and the denominator by $\sqrt{2}$. This process is called **rationalizing the denominator**.

You can use square roots to solve some types of quadratic equations. For instance, if $s > 0$, then the quadratic equation $x^2 = s$ has two real-number solutions: $x = \sqrt{s}$ and $x = -\sqrt{s}$. These solutions are often written in condensed form as $x = \pm\sqrt{s}$. The symbol $\pm\sqrt{s}$ is read as "plus or minus the square root of s."

EXAMPLE 2 *Solving a Quadratic Equation*

Solve $2x^2 + 1 = 17$.

SOLUTION

Begin by writing the equation in the form $x^2 = s$.

$2x^2 + 1 = 17$ — **Write original equation.**

$2x^2 = 16$ — **Subtract 1 from each side.**

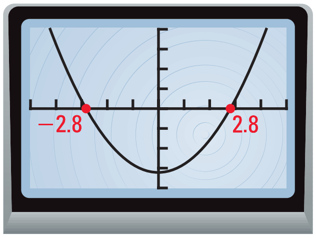
$x^2 = 8$ — **Divide each side by 2.**

$x = \pm\sqrt{8}$ — **Take square roots of each side.**

$x = \pm 2\sqrt{2}$ — **Simplify.**

▶ The solutions are $2\sqrt{2}$ and $-2\sqrt{2}$.

✓**CHECK** You can check the solutions algebraically by substituting them into the original equation. Since this equation is equivalent to $2x^2 - 16 = 0$, you can also check the solutions by graphing $y = 2x^2 - 16$ and observing that the graph's x-intercepts appear to be about $2.8 \approx 2\sqrt{2}$ and $-2.8 \approx -2\sqrt{2}$.



EXAMPLE 3 *Solving a Quadratic Equation*

Solve $\frac{1}{3}(x + 5)^2 = 7$.

SOLUTION

$\frac{1}{3}(x + 5)^2 = 7$ — **Write original equation.**

$(x + 5)^2 = 21$ — **Multiply each side by 3.**

$x + 5 = \pm\sqrt{21}$ — **Take square roots of each side.**

$x = -5 \pm \sqrt{21}$ — **Subtract 5 from each side.**

▶ The solutions are $-5 + \sqrt{21}$ and $-5 - \sqrt{21}$.

✓**CHECK** Check the solutions either by substituting them into the original equation or by graphing $y = \frac{1}{3}(x + 5)^2 - 7$ and observing the x-intercepts.

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING QUADRATIC MODELS IN REAL LIFE

When an object is dropped, its speed continually increases, and therefore its height above the ground decreases at a faster and faster rate. The height h (in feet) of the object t seconds after it is dropped can be modeled by the function

$$h = -16t^2 + h_0$$

where h_0 is the object's initial height. This model assumes that the force of air resistance on the object is negligible. Also, the model works only on Earth. For planets with stronger or weaker gravity, different models are used (see Exercise 71).

Real Life Movies

EXAMPLE 4 *Modeling a Falling Object's Height with a Quadratic Function*

A stunt man working on the set of a movie is to fall out of a window 100 feet above the ground. For the stunt man's safety, an air cushion 26 feet wide by 30 feet long by 9 feet high is positioned on the ground below the window.

a. For how many seconds will the stunt man fall before he reaches the cushion?

b. A movie camera operating at a speed of 24 frames per second records the stunt man's fall. How many frames of film show the stunt man falling?

SOLUTION

a. The stunt man's initial height is $h_0 = 100$ feet, so his height as a function of time is given by $h = -16t^2 + 100$. Since the top of the cushion is 9 feet above the ground, you can determine how long it takes the stunt man to reach the cushion by finding the value of t for which $h = 9$. Here are two methods:

Method 1: Make a table of values.

t	0	1	2	3
h	100	84	36	−44

▶ From the table you can see that $h = 9$ at a value of t between $t = 2$ and $t = 3$. It takes between 2 sec and 3 sec for the stunt man to reach the cushion.

Method 2: Solve a quadratic equation.

$h = -16t^2 + 100$ — **Write height function.**

$9 = -16t^2 + 100$ — **Substitute 9 for *h*.**

$-91 = -16t^2$ — **Subtract 100 from each side.**

$\frac{91}{16} = t^2$ — **Divide each side by –16.**

$\sqrt{\frac{91}{16}} = t$ — **Take positive square root.**

$2.4 \approx t$ — **Use a calculator.**

▶ It takes about 2.4 seconds for the stunt man to reach the cushion.

b. The number of frames of film that show the stunt man falling is given by the product (2.4 sec)(24 frames/sec), or about 57 frames.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain what it means to "rationalize the denominator" of a quotient containing square roots.

Concept Check ✓

2. State the product and quotient properties of square roots in words.

3. How many real-number solutions does the equation $x^2 = s$ have when $s > 0$? when $s = 0$? when $s < 0$?

Skill Check ✓

Simplify the expression.

4. $\sqrt{49}$ **5.** $\sqrt{12}$ **6.** $\sqrt{45}$ **7.** $\sqrt{3} \cdot \sqrt{27}$

8. $\sqrt{\frac{16}{25}}$ **9.** $\sqrt{\frac{7}{9}}$ **10.** $\frac{1}{\sqrt{3}}$ **11.** $\sqrt{\frac{5}{2}}$

Solve the equation.

12. $x^2 = 64$ **13.** $x^2 - 9 = 16$ **14.** $4x^2 + 7 = 23$

15. $\frac{x^2}{6} - 2 = 0$ **16.** $5(x - 1)^2 = 50$ **17.** $\frac{1}{2}(x + 8)^2 = 14$

18. **ENGINEERING** At an engineering school, students are challenged to design a container that prevents an egg from breaking when dropped from a height of 50 feet. Write an equation giving a container's height h (in feet) above the ground after t seconds. How long does the container take to hit the ground?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 946.

USING THE PRODUCT PROPERTY **Simplify the expression.**

19. $\sqrt{18}$ **20.** $\sqrt{48}$ **21.** $\sqrt{27}$ **22.** $\sqrt{52}$

23. $\sqrt{72}$ **24.** $\sqrt{175}$ **25.** $\sqrt{98}$ **26.** $\sqrt{605}$

27. $2\sqrt{7} \cdot \sqrt{7}$ **28.** $\sqrt{8} \cdot \sqrt{2}$ **29.** $\sqrt{3} \cdot \sqrt{12}$ **30.** $3\sqrt{20} \cdot 6\sqrt{5}$

31. $\sqrt{12} \cdot \sqrt{2}$ **32.** $\sqrt{6} \cdot \sqrt{10}$ **33.** $4\sqrt{3} \cdot \sqrt{21}$ **34.** $\sqrt{8} \cdot \sqrt{6} \cdot \sqrt{3}$

USING THE QUOTIENT PROPERTY **Simplify the expression.**

35. $\sqrt{\frac{1}{9}}$ **36.** $\sqrt{\frac{4}{49}}$ **37.** $\sqrt{\frac{36}{25}}$ **38.** $\sqrt{\frac{100}{81}}$

39. $\sqrt{\frac{3}{16}}$ **40.** $\sqrt{\frac{11}{64}}$ **41.** $\sqrt{\frac{75}{36}}$ **42.** $\sqrt{\frac{40}{169}}$

43. $\frac{2}{\sqrt{3}}$ **44.** $\frac{5}{\sqrt{17}}$ **45.** $\sqrt{\frac{6}{5}}$ **46.** $\sqrt{\frac{144}{11}}$

47. $\sqrt{\frac{7}{8}}$ **48.** $\sqrt{\frac{18}{13}}$ **49.** $\sqrt{\frac{45}{32}}$ **50.** $\sqrt{\frac{15}{7}} \cdot \sqrt{\frac{4}{3}}$

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 19–50
Example 2: Exs. 51–59
Example 3: Exs. 60–68
Example 4: Exs. 69–73

SOLVING QUADRATIC EQUATIONS **Solve the equation.**

51. $x^2 = 121$ **52.** $x^2 = 90$ **53.** $3x^2 = 108$

54. $2x^2 + 5 = 41$ **55.** $-x^2 - 12 = -87$ **56.** $7 - 10u^2 = 1$

57. $\frac{v^2}{25} - 1 = 11$ **58.** $6 - \frac{p^2}{8} = -4$ **59.** $\frac{5q^2}{6} - \frac{q^2}{3} = 72$

SOLVING QUADRATIC EQUATIONS **Solve the equation.**

60. $2(x - 3)^2 = 8$ **61.** $4(x + 1)^2 = 100$ **62.** $-3(x + 2)^2 = -18$

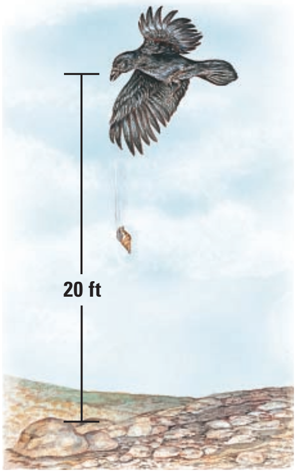
63. $5(x - 7)^2 = 135$ **64.** $8(x + 4)^2 = 9$ **65.** $2(a - 6)^2 - 45 = 53$

66. $\frac{1}{4}(b - 8)^2 = 7$ **67.** $(2r - 5)^2 = 81$ **68.** $\frac{(s + 1)^2}{10} - \frac{12}{5} = \frac{15}{2}$

69. **HISTORY CONNECTION** According to legend, in 1589 the Italian scientist Galileo Galilei dropped two rocks of different weights from the top of the Leaning Tower of Pisa. He wanted to show that the rocks would hit the ground at the same time. Given that the tower's height is about 177 feet, how long would it have taken for the rocks to hit the ground?

70. **ORNITHOLOGY** Many birds drop shellfish onto rocks to break the shell and get to the food inside. Crows along the west coast of Canada use this technique to eat whelks (a type of sea snail). Suppose a crow drops a whelk from a height of 20 feet, as shown.

▶ Source: *Cambridge Encyclopedia of Ornithology*



a. Write an equation giving the whelk's height h (in feet) after t seconds.

b. Use the *Table* feature of a graphing calculator to find h when $t = 0, 0.1, 0.2, 0.3, \ldots, 1.4, 1.5$. (You'll need to scroll down the table to see all the values.) To the nearest tenth of a second, how long does it take for the whelk to hit the ground? Check your answer by solving a quadratic equation.

FOCUS ON APPLICATIONS

ASTRONOMY The acceleration due to gravity on the moon is about 5.3 ft/sec^2. This means that the moon's gravity is only about one sixth as strong as Earth's.

APPLICATION LINK
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71. **ASTRONOMY** On *any* planet, the height h (in feet) of a falling object t seconds after it is dropped can be modeled by

$$h = -\frac{g}{2}t^2 + h_0$$

where h_0 is the object's initial height and g is the acceleration (in feet per second squared) due to the planet's gravity. For each planet in the table, find the time it takes for a rock dropped from a height of 200 feet to hit the ground.

Planet	Earth	Mars	Jupiter	Neptune	Pluto
g (ft/sec^2)	32	12	81	36	2.1

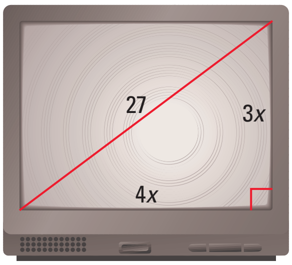
▶ Source: STARLab, Stanford University

72. **OCEANOGRAPHY** The equation $h = 0.019s^2$ gives the height h (in feet) of the largest ocean waves when the wind speed is s knots. How fast is the wind blowing if the largest waves are 15 feet high? ▶ Source: *Encyclopaedia Britannica*

STUDENT HELP

Skills Review
For help with the Pythagorean theorem, see p. 917.

73. **TELEVISION** The *aspect ratio* of a TV screen is the ratio of the screen's width to its height. For most TVs, the aspect ratio is 4:3. What are the width and height of the screen for a 27 inch TV? (*Hint:* Use the Pythagorean theorem and the fact that TV sizes such as 27 inches refer to the length of the screen's diagonal.)



Test Preparation

74. MULTI-STEP PROBLEM Building codes often require that buildings be able to withstand a certain amount of wind pressure. The pressure P (in pounds per square foot) from wind blowing at s miles per hour is given by $P = 0.00256s^2$.

▶ Source: *The Complete How to Figure It*

a. You are designing a two-story library. Buildings this tall are often required to withstand wind pressure of 20 lb/ft^2. Under this requirement, how fast can the wind be blowing before it produces excessive stress on a building?

b. To be safe, you design your library so that it can withstand wind pressure of 40 lb/ft^2. Does this mean that the library can survive wind blowing at twice the speed you found in part (a)? Justify your answer mathematically.

c. *Writing* Use the pressure formula to explain why even a relatively small increase in wind speed could have potentially serious effects on a building.

★ Challenge

75. SCIENCE CONNECTION For a bathtub with a rectangular base, *Torricelli's law* implies that the height h of water in the tub t seconds after it begins draining is given by

$$h = \left(\sqrt{h_0} - \frac{2\pi d^2\sqrt{3}}{lw}t\right)^2$$

where l and w are the tub's length and width, d is the diameter of the drain, and h_0 is the water's initial height. (All measurements are in inches.) Suppose you completely fill a tub with water. The tub is 60 inches long by 30 inches wide by 25 inches high and has a drain with a 2 inch diameter.

a. Find the time it takes for the tub to go from being full to half-full.

b. Find the time it takes for the tub to go from being half-full to empty.

c. CRITICAL THINKING Based on your results, what general statement can you make about the speed at which water drains?

EXTRA CHALLENGE
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MIXED REVIEW

SOLVING SYSTEMS Solve the linear system by graphing. (Review 3.1)

76. $x + y = 5$
$-x + 2y = 4$

77. $x - y = -1$
$3x + y = 5$

78. $-3x + y = 7$
$2x + y = 2$

79. $2x - 3y = 9$
$4x - 3y = 3$

80. $x + 4y = 4$
$3x - 2y = 12$

81. $2x + 3y = 6$
$x - 6y = 18$

MATRIX OPERATIONS Perform the indicated operation(s). (Review 4.1)

82. $\begin{bmatrix} 6 & -1 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} -5 & -4 \\ 10 & -2 \end{bmatrix}$

83. $\begin{bmatrix} 7 & 3 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} -6 & 4 \\ 9 & -1 \end{bmatrix}$

84. $-4\begin{bmatrix} -3 & 5 & -1 \\ 4 & -4 & 8 \end{bmatrix}$

85. $-2\begin{bmatrix} 12 & 10 \\ 20 & -9 \end{bmatrix} + 7\begin{bmatrix} 15 & 11 \\ 0 & -7 \end{bmatrix}$

WRITING IN STANDARD FORM Write the quadratic function in standard form. (Review 5.1 for 5.4)

86. $y = (x + 5)(x - 2)$

87. $y = (x - 1)(x - 8)$

88. $y = (2x + 7)(x + 4)$

89. $y = (4x + 9)(4x - 9)$

90. $y = (x - 3)^2 + 1$

91. $y = 5(x + 6)^2 - 12$

QUIZ 1

Self-Test for Lessons 5.1–5.3

Graph the function. **(Lesson 5.1)**

1. $y = x^2 - 2x - 3$ **2.** $y = 2(x + 2)^2 + 1$ **3.** $y = -\frac{1}{3}(x + 5)(x - 1)$

Solve the equation. **(Lesson 5.2)**

4. $x^2 - 6x - 27 = 0$ **5.** $4x^2 + 21x + 20 = 0$ **6.** $7t^2 - 4t = 3t^2 - 1$

Simplify the expression. **(Lesson 5.3)**

7. $\sqrt{54}$ **8.** $7\sqrt{2} \cdot \sqrt{10}$ **9.** $\sqrt{\frac{36}{5}}$ **10.** $\frac{4}{\sqrt{12}}$

11. **SWIMMING** The drag force F (in pounds) of water on a swimmer can be modeled by $F = 1.35s^2$ where s is the swimmer's speed (in miles per hour). How fast must you swim to generate a drag force of 10 pounds? **(Lesson 5.3)**

MATH & History — Telescopes

APPLICATION LINK www.mcdougallittell.com

THEN

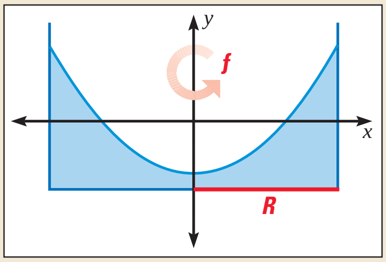
THE FIRST TELESCOPE is thought to have been made in 1608 by Hans Lippershey, a Dutch optician. Lippershey's telescope, called a *refracting telescope*, used lenses to magnify objects. Another type of telescope is a *reflecting telescope*. Reflecting telescopes magnify objects with parabolic mirrors, traditionally made from glass.

NOW

RECENTLY "liquid mirrors" for telescopes have been made by spinning reflective liquids, such as mercury. A cross section of the surface of a spinning liquid is a parabola with equation

$$y = \frac{\pi^2 f^2}{16}x^2 - \frac{\pi^2 f^2 R^2}{32}$$

where f is the spinning frequency (in revolutions per second) and R is the radius (in feet) of the container.



1. Write an equation for the surface of a liquid before it is spun. What does the equation tell you about the location of the x-axis relative to the liquid?

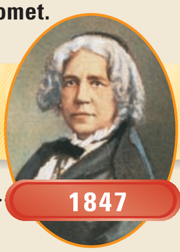
2. Suppose mercury is spun with a frequency of 0.5 revolution/sec in a container with radius 2 feet. Write and graph an equation for the mercury's surface.

3. Find the x-intercepts of the graph of $y = \frac{\pi^2 f^2}{16}x^2 - \frac{\pi^2 f^2 R^2}{32}$. Does changing the spinning frequency affect the x-intercepts? Explain.

1609 Galileo first uses a refracting telescope for astronomical purposes.

1668 Isaac Newton builds first reflecting telescope.

1847 Maria Mitchell is first to use a telescope to discover a comet.



1987 Liquid mirrors are first used to do astronomical research.