

5.2

Solving Quadratic Equations by Factoring

What you should learn

GOAL 1 Factor quadratic expressions and solve quadratic equations by factoring.

GOAL 2 Find zeros of quadratic functions, as applied in Example 8.

Why you should learn it

▼ To solve real-life problems, such as finding appropriate dimensions for a mural in Ex. 97.



GOAL 1 FACTORING QUADRATIC EXPRESSIONS

You know how to write $(x + 3)(x + 5)$ as $x^2 + 8x + 15$. The expressions $x + 3$ and $x + 5$ are **binomials** because they have two terms. The expression $x^2 + 8x + 15$ is a **trinomial** because it has three terms. You can use **factoring** to write a trinomial as a product of binomials. To factor $x^2 + bx + c$, find integers m and n such that:

$$\begin{aligned} x^2 + bx + c &= (x + m)(x + n) \\ &= x^2 + (m + n)x + mn \end{aligned}$$

So, the *sum* of m and n must equal b and the *product* of m and n must equal c .

EXAMPLE 1 Factoring a Trinomial of the Form $x^2 + bx + c$

Factor $x^2 - 12x - 28$.

SOLUTION

You want $x^2 - 12x - 28 = (x + m)(x + n)$ where $mn = -28$ and $m + n = -12$.

Factors of -28 (m, n)	$-1, 28$	$1, -28$	$-2, 14$	$2, -14$	$-4, 7$	$4, -7$
Sum of factors ($m + n$)	27	-27	12	-12	3	-3

► The table shows that $m = 2$ and $n = -14$. So, $x^2 - 12x - 28 = (x + 2)(x - 14)$.

To factor $ax^2 + bx + c$ when $a \neq 1$, find integers $k, l, m,$ and n such that:

$$\begin{aligned} ax^2 + bx + c &= (kx + m)(lx + n) \\ &= klx^2 + (kn + lm)x + mn \end{aligned}$$

Therefore, k and l must be factors of a , and m and n must be factors of c .

EXAMPLE 2 Factoring a Trinomial of the Form $ax^2 + bx + c$

Factor $3x^2 - 17x + 10$.

SOLUTION

You want $3x^2 - 17x + 10 = (kx + m)(lx + n)$ where k and l are factors of 3 and m and n are (negative) factors of 10. Check possible factorizations by multiplying.

$$(3x - 10)(x - 1) = 3x^2 - 13x + 10 \quad (3x - 1)(x - 10) = 3x^2 - 31x + 10$$

$$(3x - 5)(x - 2) = 3x^2 - 11x + 10 \quad (3x - 2)(x - 5) = 3x^2 - 17x + 10 \checkmark$$

► The correct factorization is $3x^2 - 17x + 10 = (3x - 2)(x - 5)$.

STUDENT HELP

Skills Review

For help with factoring, see p. 938.

As in Example 2, factoring quadratic expressions often involves trial and error. However, some expressions are easy to factor because they follow special patterns.

SPECIAL FACTORING PATTERNS

PATTERN NAME	PATTERN	EXAMPLE
Difference of Two Squares	$a^2 - b^2 = (a + b)(a - b)$	$x^2 - 9 = (x + 3)(x - 3)$
Perfect Square Trinomial	$a^2 + 2ab + b^2 = (a + b)^2$	$x^2 + 12x + 36 = (x + 6)^2$
	$a^2 - 2ab + b^2 = (a - b)^2$	$x^2 - 8x + 16 = (x - 4)^2$

EXAMPLE 3 Factoring with Special Patterns

Factor the quadratic expression.

a. $4x^2 - 25 = (2x)^2 - 5^2$ Difference of two squares
 $= (2x + 5)(2x - 5)$

b. $9y^2 + 24y + 16 = (3y)^2 + 2(3y)(4) + 4^2$ Perfect square trinomial
 $= (3y + 4)^2$

c. $49r^2 - 14r + 1 = (7r)^2 - 2(7r)(1) + 1^2$ Perfect square trinomial
 $= (7r - 1)^2$

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A **monomial** is an expression that has only one term. As a first step to factoring, you should check to see whether the terms have a common monomial factor.

EXAMPLE 4 Factoring Monomials First

Factor the quadratic expression.

a. $5x^2 - 20 = 5(x^2 - 4)$ b. $6p^2 + 15p + 9 = 3(2p^2 + 5p + 3)$
 $= 5(x + 2)(x - 2)$ $= 3(2p + 3)(p + 1)$

c. $2u^2 + 8u = 2u(u + 4)$ d. $4x^2 + 4x - 4 = 4(x^2 + x - 1)$

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You can use factoring to solve certain *quadratic equations*. A **quadratic equation** in one variable can be written in the form $ax^2 + bx + c = 0$ where $a \neq 0$. This is called the **standard form** of the equation. If the left side of $ax^2 + bx + c = 0$ can be factored, then the equation can be solved using the *zero product property*.

STUDENT HELP

Study Tip

It is not always possible to factor a trinomial into a product of two binomials with integer coefficients. For instance, the trinomial $x^2 + x - 1$ in part (d) of Example 4 cannot be factored. Such trinomials are called *irreducible*.

ZERO PRODUCT PROPERTY

Let A and B be real numbers or algebraic expressions. If $AB = 0$, then $A = 0$ or $B = 0$.

EXAMPLE 5 Solving Quadratic Equations

Solve (a) $x^2 + 3x - 18 = 0$ and (b) $2t^2 - 17t + 45 = 3t - 5$.

SOLUTION

a. $x^2 + 3x - 18 = 0$

Write original equation.

$$(x + 6)(x - 3) = 0$$

Factor.

$$x + 6 = 0 \quad \text{or} \quad x - 3 = 0$$

Use zero product property.

$$x = -6 \quad \text{or} \quad x = 3$$

Solve for x .

▶ The solutions are -6 and 3 . Check the solutions in the original equation.

b. $2t^2 - 17t + 45 = 3t - 5$

Write original equation.

$$2t^2 - 20t + 50 = 0$$

Write in standard form.

$$t^2 - 10t + 25 = 0$$

Divide each side by 2.

$$(t - 5)^2 = 0$$

Factor.

$$t - 5 = 0$$

Use zero product property.

$$t = 5$$

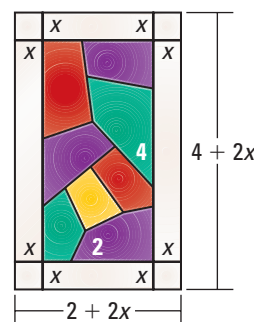
Solve for t .

▶ The solution is 5 . Check the solution in the original equation.



EXAMPLE 6 Using a Quadratic Equation as a Model

You have made a rectangular stained glass window that is 2 feet by 4 feet. You have 7 square feet of clear glass to create a border of uniform width around the window. What should the width of the border be?



SOLUTION

PROBLEM SOLVING STRATEGY

VERBAL MODEL

$$\boxed{\text{Area of border}} = \boxed{\text{Area of border and window}} - \boxed{\text{Area of window}}$$

LABELS

Width of border = x (feet)
 Area of border = 7 (square feet)
 Area of border and window = $(2 + 2x)(4 + 2x)$ (square feet)
 Area of window = $2 \cdot 4 = 8$ (square feet)

ALGEBRAIC MODEL

$$7 = (2 + 2x)(4 + 2x) - 8$$

Write algebraic model.

$$0 = 4x^2 + 12x - 7$$

Write in standard form.

$$0 = (2x + 7)(2x - 1)$$

Factor.

$$2x + 7 = 0 \quad \text{or} \quad 2x - 1 = 0$$

Use zero product property.

$$x = -3.5 \quad \text{or} \quad x = 0.5$$

Solve for x .

▶ Reject the negative value, -3.5 . The border's width should be 0.5 ft, or 6 in.

GOAL 2 FINDING ZEROS OF QUADRATIC FUNCTIONS

In Lesson 5.1 you learned that the x -intercepts of the graph of $y = a(x - p)(x - q)$ are p and q . The numbers p and q are also called **zeros** of the function because the function's value is zero when $x = p$ and when $x = q$. If a quadratic function is given in standard form $y = ax^2 + bx + c$, you may be able to find its zeros by using factoring to rewrite the function in intercept form.

EXAMPLE 7 Finding the Zeros of a Quadratic Function

Find the zeros of $y = x^2 - x - 6$.

STUDENT HELP

Study Tip

In Example 7 note that -2 and 3 are zeros of the function and x -intercepts of the graph. In general, functions have zeros and graphs have x -intercepts.

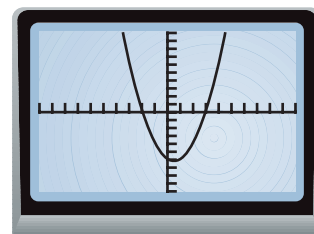
SOLUTION

Use factoring to write the function in intercept form.

$$\begin{aligned}y &= x^2 - x - 6 \\ &= (x + 2)(x - 3)\end{aligned}$$

▶ The zeros of the function are -2 and 3 .

✓ **CHECK** Graph $y = x^2 - x - 6$. The graph passes through $(-2, 0)$ and $(3, 0)$, so the zeros are -2 and 3 .



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From Lesson 5.1 you know that the vertex of the graph of $y = a(x - p)(x - q)$ lies on the vertical line halfway between $(p, 0)$ and $(q, 0)$. In terms of zeros, the function has its maximum or minimum value when x equals the *average* of the zeros.

EXAMPLE 8 Using the Zeros of a Quadratic Model

STUDENT HELP



HOMEWORK HELP

Visit our Web site www.mcdougallittell.com for extra examples.

BUSINESS You maintain a music-oriented Web site that allows subscribing customers to download audio and video clips of their favorite bands. When the subscription price is \$16 per year, you get 30,000 subscribers. For each \$1 increase in price, you expect to lose 1000 subscribers. How much should you charge to maximize your annual revenue? What is your maximum revenue?

SOLUTION

$$\text{Revenue} = \text{Number of subscribers} \cdot \text{Subscription price}$$

Let R be your annual revenue and let x be the number of \$1 price increases.

$$\begin{aligned}R &= (30,000 - 1000x)(16 + x) \\ &= (-1000x + 30,000)(x + 16) \\ &= -1000(x - 30)(x + 16)\end{aligned}$$

The zeros of the revenue function are 30 and -16 . The value of x that maximizes R is the average of the zeros, or $x = \frac{30 + (-16)}{2} = 7$.

▶ To maximize revenue, charge $\$16 + \$7 = \$23$ per year for a subscription. Your maximum revenue is $R = -1000(7 - 30)(7 + 16) = \$529,000$.

GUIDED PRACTICE

Vocabulary Check ✓

1. What is a zero of a function $y = f(x)$?

Concept Check ✓

2. In Example 2, how do you know that m and n must be *negative* factors of 10?

3. **ERROR ANALYSIS** A student solved $x^2 + 4x + 3 = 8$ as shown. Explain the student's mistake. Then solve the equation correctly.

$$\begin{aligned} x^2 + 4x + 3 &= 8 \\ (x + 3)(x + 1) &= 8 \\ x + 3 &= 8 \text{ or } x + 1 = 8 \\ x &= 5 \text{ or } x = 7 \end{aligned}$$

Skill Check ✓

Factor the expression.

4. $x^2 - x - 2$

5. $2x^2 + x - 3$

6. $x^2 - 16$

7. $y^2 + 2y + 1$

8. $p^2 - 4p + 4$

9. $q^2 + q$

Solve the equation.

10. $(x + 3)(x - 1) = 0$

11. $x^2 - 2x - 8 = 0$

12. $3x^2 + 10x + 3 = 0$

13. $4u^2 - 1 = 0$

14. $v^2 - 14v = -49$

15. $5w^2 = 30w$

Write the quadratic function in intercept form and give the function's zeros.

16. $y = x^2 - 6x + 5$

17. $y = x^2 + 6x + 8$

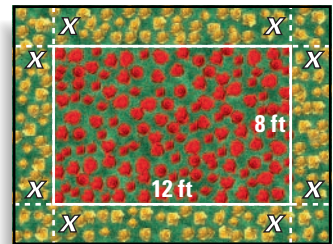
18. $y = x^2 - 1$

19. $y = x^2 + 10x + 25$

20. $y = 2x^2 - 2x - 24$

21. $y = 3x^2 - 8x + 4$

22. **URBAN PLANNING** You have just planted a rectangular flower bed of red roses in a park near your home. You want to plant a border of yellow roses around the flower bed as shown. Since you bought the same number of red and yellow roses, the areas of the border and inner flower bed will be equal. What should the width x of the border be?



PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 945.

FACTORIZING $x^2 + bx + c$ Factor the trinomial. If the trinomial cannot be factored, say so.

23. $x^2 + 5x + 4$

24. $x^2 + 9x + 14$

25. $x^2 + 13x + 40$

26. $x^2 - 4x + 3$

27. $x^2 - 8x + 12$

28. $x^2 - 16x + 51$

29. $a^2 + 3a - 10$

30. $b^2 + 6b - 27$

31. $c^2 + 2c - 80$

32. $p^2 - 5p - 6$

33. $q^2 - 7q - 10$

34. $r^2 - 14r - 72$

FACTORIZING $ax^2 + bx + c$ Factor the trinomial. If the trinomial cannot be factored, say so.

35. $2x^2 + 7x + 3$

36. $3x^2 + 17x + 10$

37. $8x^2 + 18x + 9$

38. $5x^2 - 7x + 2$

39. $6x^2 - 9x + 5$

40. $10x^2 - 19x + 6$

41. $3k^2 + 32k - 11$

42. $11m^2 + 14m - 16$

43. $18n^2 + 9n - 14$

44. $7u^2 - 4u - 3$

45. $12v^2 - 25v - 7$

46. $4w^2 - 13w - 27$

STUDENT HELP

HOMEWORK HELP

- Example 1:** Exs. 23–34
Example 2: Exs. 35–46
Example 3: Exs. 47–55
Example 4: Exs. 56–64
Example 5: Exs. 65–79
Example 6: Exs. 90, 91, 97, 98
Example 7: Exs. 80–88
Example 8: Exs. 99–101

FACTORING WITH SPECIAL PATTERNS Factor the expression.

- | | | |
|---------------------|------------------------|--------------------------|
| 47. $x^2 - 25$ | 48. $x^2 + 4x + 4$ | 49. $x^2 - 6x + 9$ |
| 50. $4r^2 - 4r + 1$ | 51. $9s^2 + 12s + 4$ | 52. $16t^2 - 9$ |
| 53. $49 - 100a^2$ | 54. $25b^2 - 60b + 36$ | 55. $81c^2 + 198c + 121$ |

FACTORING MONOMIALS FIRST Factor the expression.

- | | | |
|-----------------------|--------------------------|------------------------|
| 56. $5x^2 + 5x - 10$ | 57. $18x^2 - 2$ | 58. $3x^2 + 54x + 243$ |
| 59. $8y^2 - 28y - 60$ | 60. $112a^2 - 168a + 63$ | 61. $u^2 + 7u$ |
| 62. $6t^2 - 36t$ | 63. $-v^2 + 2v - 1$ | 64. $2d^2 + 12d - 16$ |

EQUATIONS IN STANDARD FORM Solve the equation.

- | | | |
|-------------------------|---------------------------|---------------------------|
| 65. $x^2 - 3x - 4 = 0$ | 66. $x^2 + 19x + 88 = 0$ | 67. $5x^2 - 13x + 6 = 0$ |
| 68. $8x^2 - 6x - 5 = 0$ | 69. $k^2 + 24k + 144 = 0$ | 70. $9m^2 - 30m + 25 = 0$ |
| 71. $81n^2 - 16 = 0$ | 72. $40a^2 + 4a = 0$ | 73. $-3b^2 + 3b + 90 = 0$ |

EQUATIONS NOT IN STANDARD FORM Solve the equation.

- | | |
|-------------------------------------|-----------------------------------|
| 74. $x^2 + 9x = -20$ | 75. $16x^2 = 8x - 1$ |
| 76. $5p^2 - 25 = 4p^2 + 24$ | 77. $2y^2 - 4y - 8 = -y^2 + y$ |
| 78. $2q^2 + 4q - 1 = 7q^2 - 7q + 1$ | 79. $(w + 6)^2 = 3(w + 12) - w^2$ |

FINDING ZEROS Write the quadratic function in intercept form and give the function's zeros.

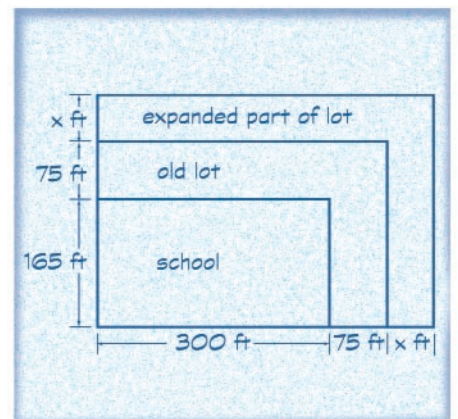
- | | | |
|---------------------------|---------------------------|-------------------------|
| 80. $y = x^2 - 3x + 2$ | 81. $y = x^2 + 7x + 12$ | 82. $y = x^2 + 2x - 35$ |
| 83. $y = x^2 - 4$ | 84. $y = x^2 + 20x + 100$ | 85. $y = x^2 - 3x$ |
| 86. $y = 3x^2 - 12x - 15$ | 87. $y = -x^2 + 16x - 64$ | 88. $y = 2x^2 - 9x + 4$ |

89. LOGICAL REASONING Is there a formula for factoring the *sum* of two squares? You will investigate this question in parts (a) and (b).

- Consider the sum of squares $x^2 + 9$. If this sum can be factored, then there are integers m and n such that $x^2 + 9 = (x + m)(x + n)$. Write two equations relating the sum and the product of m and n to the coefficients in $x^2 + 9$.
- Show that there are no integers m and n that satisfy both equations you wrote in part (a). What can you conclude?

90. QUILTING You have made a quilt that is 4 feet by 5 feet. You want to use the remaining 10 square feet of fabric to add a decorative border of uniform width. What should the width of the border be?

91. CONSTRUCTION A high school wants to double the size of its parking lot by expanding the existing lot as shown. By what distance x should the lot be expanded?

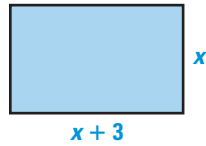


STUDENT HELP

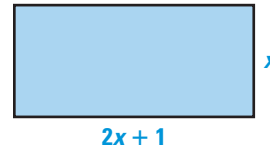
Skills Review
For help with areas of geometric figures, see p. 914.

GEOMETRY CONNECTION Find the value of x .

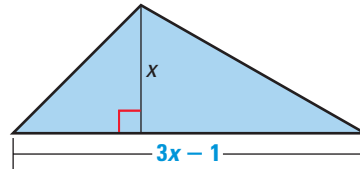
92. Area of rectangle = 40



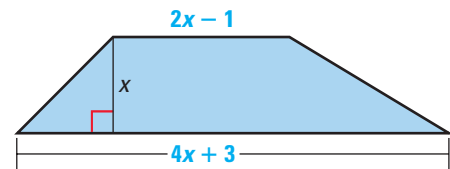
93. Area of rectangle = 105



94. Area of triangle = 22



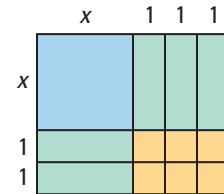
95. Area of trapezoid = 114



96. **VISUAL THINKING** Use the diagram shown at the right.

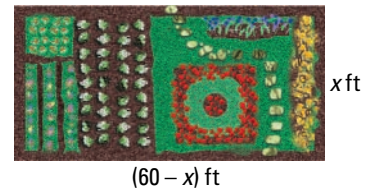
a. Explain how the diagram models the factorization $x^2 + 5x + 6 = (x + 2)(x + 3)$.

b. Draw a diagram that models the factorization $x^2 + 7x + 12 = (x + 3)(x + 4)$.



97. **ART CONNECTION** As part of Black History Month in February, an artist is creating a mural on the side of a building. A painting of Dr. Martin Luther King, Jr., will occupy the center of the mural and will be surrounded by a border of uniform width showing other prominent African-Americans. The side of the building is 50 feet wide by 30 feet high, and the artist wants to devote 25% of the available space to the border. What should the width of the border be?

98. **ENVIRONMENT** A student environmental group wants to build an ecology garden as shown. The area of the garden should be 800 square feet to accommodate all the species of plants the group wants to grow. A construction company has donated 120 feet of iron fencing to enclose the garden. What should the dimensions of the garden be?



99. **ATHLETIC WEAR** A shoe store sells about 200 pairs of a new basketball shoe each month when it charges \$60 per pair. For each \$1 increase in price, about 2 fewer pairs per month are sold. How much per pair should the store charge to maximize monthly revenue? What is the maximum revenue?

100. **HOME ELECTRONICS** The manager of a home electronics store is considering repricing a new model of digital camera. At the current price of \$680, the store sells about 70 cameras each month. Sales data from other stores indicate that for each \$20 decrease in price, about 5 more cameras per month would be sold. How much should the manager charge for a camera to maximize monthly revenue? What is the maximum revenue?

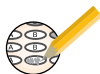
101. **HISTORY CONNECTION** Big Bertha, a cannon used in World War I, could fire shells incredibly long distances. The path of a shell could be modeled by $y = -0.0196x^2 + 1.37x$ where x was the horizontal distance traveled (in miles) and y was the height (in miles). How far could Big Bertha fire a shell? What was the shell's maximum height?
▶ Source: World War I: Trenches on the Web

FOCUS ON APPLICATIONS



REAL LIFE ENVIRONMENT
Ecology gardens are often used to conduct research with different plant species under a variety of growing conditions.

Test Preparation



102. MULTIPLE CHOICE Suppose $x^2 + 4x + c = (x + m)(x + n)$ where c , m , and n are integers. Which of the following are *not* possible values of m and n ?

- (A) $m = 2, n = 2$ (B) $m = -1, n = 5$
 (C) $m = -2, n = -2$ (D) $m = 1, n = 3$

103. MULTIPLE CHOICE What are all solutions of $2x^2 - 11x + 16 = x^2 - 3x$?

- (A) 2, 6 (B) -4 (C) -4, 4 (D) 4

104. MULTIPLE CHOICE Given that 4 is a zero of $y = 3x^2 + bx - 8$, what is the value of b ?

- (A) -40 (B) -10 (C) -8 (D) 2

★ Challenge

105. MULTICULTURAL MATHEMATICS The following problem is from the *Chiu chang suan shu*, an ancient Chinese mathematics text. Solve the problem. (*Hint*: Use the Pythagorean theorem.)

A rod of unknown length is used to measure the dimensions of a rectangular door. The rod is 4 ch'ih longer than the width of the door, 2 ch'ih longer than the height of the door, and the same length as the door's diagonal. What are the dimensions of the door? (Note: 1 ch'ih is slightly greater than 1 foot.)

EXTRA CHALLENGE

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MIXED REVIEW

ABSOLUTE VALUE Solve the equation or inequality. (Review 1.7)

106. $|x| = 3$ 107. $|x - 2| = 6$ 108. $|4x - 9| = 2$
 109. $|-5x + 4| = 14$ 110. $|7 - 3x| = -8$ 111. $|x + 1| < 3$
 112. $|2x - 5| \leq 1$ 113. $|x - 4| > 7$ 114. $|\frac{1}{3}x + 1| \geq 2$

GRAPHING LINEAR EQUATIONS Graph the equation. (Review 2.3)

115. $y = x + 1$ 116. $y = -2x + 3$ 117. $y = 3x - 5$
 118. $y = -\frac{5}{2}x + 7$ 119. $x + y = 4$ 120. $2x - y = 6$
 121. $3x + 4y = -12$ 122. $-5x + 3y = 15$ 123. $y = 2$
 124. $y = -3$ 125. $x = -1$ 126. $x = 4$

GRAPHING QUADRATIC FUNCTIONS Graph the function. (Review 5.1 for 5.3)

127. $y = x^2 - 2$ 128. $y = 2x^2 - 5$ 129. $y = -x^2 + 3$
 130. $y = (x + 1)^2 - 4$ 131. $y = -(x - 2)^2 + 1$ 132. $y = -3(x + 3)^2 + 7$
 133. $y = \frac{1}{4}x^2 - 1$ 134. $y = \frac{1}{2}(x - 4)^2 - 6$ 135. $y = -\frac{2}{3}(x + 1)(x - 3)$

136. COMMUTING You can take either the subway or the bus to your after-school job. A round trip from your home to where you work costs \$2 on the subway and \$3 on the bus. You prefer to take the bus as often as possible but can afford to spend only \$50 per month on transportation. If you work 22 days each month, how many of these days can you take the bus? (Review 1.5)