5.1 Graphing Quadratic Functions

What *you should learn*

GOAL 1 Graph quadratic functions.

GOAL 2 Use quadratic functions to solve **real-life** problems, such as finding comfortable temperatures in **Example 5**.

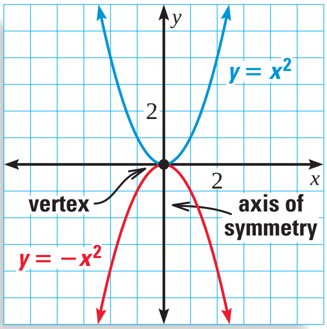
Why *you should learn it*

▼ To model **real-life** objects, such as the cables of the Golden Gate Bridge in **Example 6**.

REAL LIFE

GOAL 1 GRAPHING A QUADRATIC FUNCTION

A **quadratic function** has the form $y = ax^2 + bx + c$ where $a \neq 0$. The graph of a quadratic function is U-shaped and is called a **parabola**.



For instance, the graphs of $y = x^2$ and $y = -x^2$ are shown at the right. The origin is the lowest point on the graph of $y = x^2$ and the highest point on the graph of $y = -x^2$. The lowest or highest point on the graph of a quadratic function is called the **vertex**.

The graphs of $y = x^2$ and $y = -x^2$ are symmetric about the y-axis, called the *axis of symmetry*. In general, the **axis of symmetry** for the graph of a quadratic function is the vertical line through the vertex.

ACTIVITY
Developing Concepts

Investigating Parabolas

1. Use a graphing calculator to graph each of these functions in the same viewing window: $y = \frac{1}{2}x^2$, $y = x^2$, $y = 2x^2$, and $y = 3x^2$.
2. Repeat **Step 1** for these functions: $y = -\frac{1}{2}x^2$, $y = -x^2$, $y = -2x^2$, and $y = -3x^2$.
3. What are the vertex and axis of symmetry of the graph of $y = ax^2$?
4. Describe the effect of a on the graph of $y = ax^2$.

In the activity you examined the graph of the simple quadratic function $y = ax^2$. The graph of the more general function $y = ax^2 + bx + c$ is described below.

CONCEPT SUMMARY — **THE GRAPH OF A QUADRATIC FUNCTION**

The graph of $y = ax^2 + bx + c$ is a parabola with these characteristics:

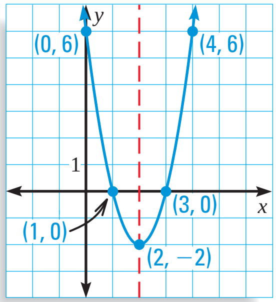
- The parabola opens up if $a > 0$ and opens down if $a < 0$. The parabola is wider than the graph of $y = x^2$ if $|a| < 1$ and narrower than the graph of $y = x^2$ if $|a| > 1$.
- The x-coordinate of the vertex is $-\frac{b}{2a}$.
- The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.

EXAMPLE 1 *Graphing a Quadratic Function*

Graph $y = 2x^2 - 8x + 6$.

SOLUTION

Note that the coefficients for this function are $a = 2$, $b = -8$, and $c = 6$. Since $a > 0$, the parabola opens up.



Find and plot the vertex. The x-coordinate is:

$$x = -\frac{b}{2a} = -\frac{-8}{2(2)} = 2$$

The y-coordinate is:

$$y = 2(2)^2 - 8(2) + 6 = -2$$

So, the vertex is $(2, -2)$.

Draw the axis of symmetry $x = 2$.

Plot two points on one side of the axis of symmetry, such as $(1, 0)$ and $(0, 6)$. Use symmetry to plot two more points, such as $(3, 0)$ and $(4, 6)$.

Draw a parabola through the plotted points.

STUDENT HELP

Skills Review
For help with symmetry, see p. 919.

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The quadratic function $y = ax^2 + bx + c$ is written in **standard form**. Two other useful forms for quadratic functions are given below.

VERTEX AND INTERCEPT FORMS OF A QUADRATIC FUNCTION

FORM OF QUADRATIC FUNCTION	CHARACTERISTICS OF GRAPH
Vertex form: $y = a(x - h)^2 + k$	The vertex is (h, k).
	The axis of symmetry is $x = h$.
Intercept form: $y = a(x - p)(x - q)$	The x-intercepts are p and q.
	The axis of symmetry is halfway between $(p, 0)$ and $(q, 0)$.

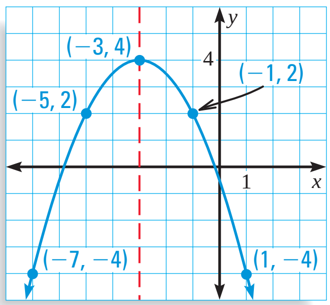
For both forms, the graph opens up if $a > 0$ and opens down if $a < 0$.

EXAMPLE 2 *Graphing a Quadratic Function in Vertex Form*

Graph $y = -\frac{1}{2}(x + 3)^2 + 4$.

SOLUTION

The function is in vertex form $y = a(x - h)^2 + k$ where $a = -\frac{1}{2}$, $h = -3$, and $k = 4$. Since $a < 0$, the parabola opens down. To graph the function, first plot the vertex $(h, k) = (-3, 4)$. Draw the axis of symmetry $x = -3$ and plot two points on one side of it, such as $(-1, 2)$ and $(1, -4)$. Use symmetry to complete the graph.



STUDENT HELP

Look Back
For help with graphing functions, see p. 123.

EXAMPLE 3 Graphing a Quadratic Function in Intercept Form

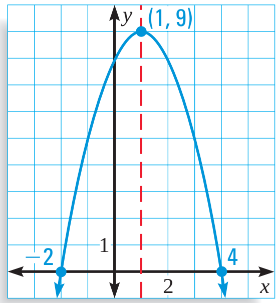
Graph $y = -(x + 2)(x - 4)$.

SOLUTION

The quadratic function is in intercept form $y = a(x - p)(x - q)$ where $a = -1$, $p = -2$, and $q = 4$. The x-intercepts occur at $(-2, 0)$ and $(4, 0)$. The axis of symmetry lies halfway between these points, at $x = 1$. So, the x-coordinate of the vertex is $x = 1$ and the y-coordinate of the vertex is:

$$y = -(1 + 2)(1 - 4) = 9$$

The graph of the function is shown.



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STUDENT HELP

Skills Review
For help with multiplying algebraic expressions, see p. 937.

You can change quadratic functions from intercept form or vertex form to standard form by multiplying algebraic expressions. One method for multiplying expressions containing two terms is *FOIL*. Using this method, you add the products of the *F*irst terms, the *O*uter terms, the *I*nner terms, and the *L*ast terms. Here is an example:

$$(x + 3)(x + 5) = \overset{F}{x^2} + \overset{O}{5x} + \overset{I}{3x} + \overset{L}{15} = x^2 + 8x + 15$$

Methods for changing from standard form to intercept form or vertex form will be discussed in Lessons 5.2 and 5.5.

EXAMPLE 4 Writing Quadratic Functions in Standard Form

Write the quadratic function in standard form.

a. $y = -(x + 4)(x - 9)$ **b.** $y = 3(x - 1)^2 + 8$

SOLUTION

a. $y = -(x + 4)(x - 9)$ **Write original function.**

$= -(x^2 - 9x + 4x - 36)$ **Multiply using FOIL.**

$= -(x^2 - 5x - 36)$ **Combine like terms.**

$= -x^2 + 5x + 36$ **Use distributive property.**

b. $y = 3(x - 1)^2 + 8$ **Write original function.**

$= 3(x - 1)(x - 1) + 8$ **Rewrite $(x - 1)^2$.**

$= 3(x^2 - x - x + 1) + 8$ **Multiply using FOIL.**

$= 3(x^2 - 2x + 1) + 8$ **Combine like terms.**

$= 3x^2 - 6x + 3 + 8$ **Use distributive property.**

$= 3x^2 - 6x + 11$ **Combine like terms.**

GOAL 2 USING QUADRATIC FUNCTIONS IN REAL LIFE

REAL LIFE
Temperature

EXAMPLE 5 *Using a Quadratic Model in Standard Form*

Researchers conducted an experiment to determine temperatures at which people feel comfortable. The percent y of test subjects who felt comfortable at temperature x (in degrees Fahrenheit) can be modeled by:

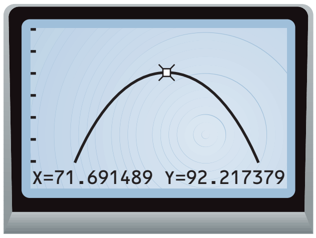
$$y = -3.678x^2 + 527.3x - 18{,}807$$

What temperature made the greatest percent of test subjects comfortable? At that temperature, what percent felt comfortable? ▶ Source: *Design with Climate*

SOLUTION

Since $a = -3.678$ is negative, the graph of the quadratic function opens down and the function has a maximum value. The maximum value occurs at:

$$x = -\frac{b}{2a} = -\frac{527.3}{2(-3.678)} \approx 72$$



The corresponding value of y is:

$$y = -3.678(72)^2 + 527.3(72) - 18{,}807 \approx 92$$

▶ The temperature that made the greatest percent of test subjects comfortable was about 72°F. At that temperature about 92% of the subjects felt comfortable.

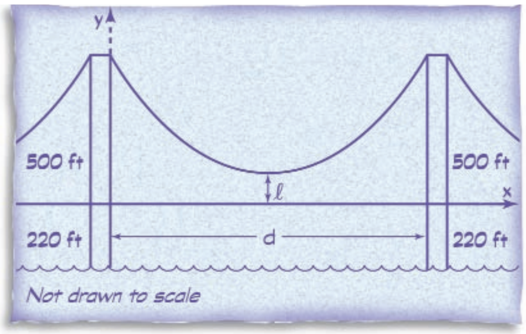
EXAMPLE 6 *Using a Quadratic Model in Vertex Form*

CIVIL ENGINEERING The Golden Gate Bridge in San Francisco has two towers that rise 500 feet above the road and are connected by suspension cables as shown. Each cable forms a parabola with equation

$$y = \frac{1}{8960}(x - 2100)^2 + 8$$

where x and y are measured in feet.

▶ Source: Golden Gate Bridge, Highway and Transportation District



a. What is the distance d between the two towers?

b. What is the height ℓ above the road of a cable at its lowest point?

FOCUS ON CAREERS

CIVIL ENGINEER Civil engineers design bridges, roads, buildings, and other structures. In 1996 civil engineers held about 196,000 jobs in the United States.

INTERNET CAREER LINK
www.mcdougallittell.com

SOLUTION

a. The vertex of the parabola is (2100, 8), so a cable's lowest point is 2100 feet from the left tower shown above. Since the heights of the two towers are the same, the symmetry of the parabola implies that the vertex is also 2100 feet from the right tower. Therefore, the towers are $d = 2(2100) = 4200$ feet apart.

b. The height ℓ above the road of a cable at its lowest point is the y-coordinate of the vertex. Since the vertex is (2100, 8), this height is $\ell = 8$ feet.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: The graph of a quadratic function is called a(n) __?__.

Concept Check ✓

2. Does the graph of $y = 3x^2 - x - 2$ open up or down? Explain.

3. Is $y = -2(x - 5)(x - 8)$ in standard form, vertex form, or intercept form?

Skill Check ✓

Graph the quadratic function. Label the vertex and axis of symmetry.

4. $y = x^2 - 4x + 7$ **5.** $y = 2(x + 1)^2 - 4$ **6.** $y = -(x + 2)(x - 1)$

7. $y = -\frac{1}{3}x^2 - 2x - 3$ **8.** $y = -\frac{3}{5}(x - 4)^2 + 6$ **9.** $y = \frac{5}{2}x(x - 3)$

Write the quadratic function in standard form.

10. $y = (x + 1)(x + 2)$ **11.** $y = -2(x + 4)(x - 3)$ **12.** $y = 4(x - 1)^2 + 5$

13. $y = -(x + 2)^2 - 7$ **14.** $y = -\frac{1}{2}(x - 6)(x - 8)$ **15.** $y = \frac{2}{3}(x - 9)^2 - 4$

16. **SCIENCE CONNECTION** The equation given in Example 5 is based on temperature preferences of both male and female test subjects. Researchers also analyzed data for males and females separately and obtained the equations below.

Males: $y = -4.290x^2 + 612.6x - 21{,}773$

Females: $y = -6.224x^2 + 908.9x - 33{,}092$

What was the most comfortable temperature for the males? for the females?

PRACTICE AND APPLICATIONS

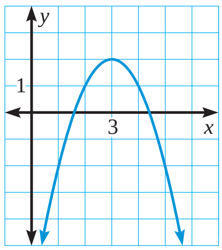
STUDENT HELP

Extra Practice to help you master skills is on p. 945.

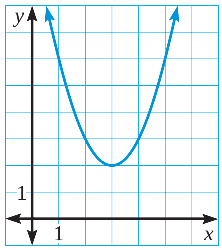
MATCHING GRAPHS **Match the quadratic function with its graph.**

17. $y = (x + 2)(x - 3)$ **18.** $y = -(x - 3)^2 + 2$ **19.** $y = x^2 - 6x + 11$

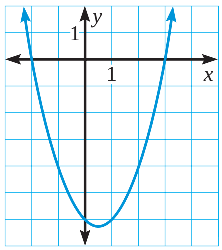
A.



B.



C.



GRAPHING WITH STANDARD FORM **Graph the quadratic function. Label the vertex and axis of symmetry.**

20. $y = x^2 - 2x - 1$ **21.** $y = 2x^2 - 12x + 19$ **22.** $y = -x^2 + 4x - 2$

23. $y = -3x^2 + 5$ **24.** $y = \frac{1}{2}x^2 + 4x + 5$ **25.** $y = -\frac{1}{6}x^2 - x - 3$

GRAPHING WITH VERTEX FORM **Graph the quadratic function. Label the vertex and axis of symmetry.**

26. $y = (x - 1)^2 + 2$ **27.** $y = -(x - 2)^2 - 1$ **28.** $y = -2(x + 3)^2 - 4$

29. $y = 3(x + 4)^2 + 5$ **30.** $y = -\frac{1}{3}(x + 1)^2 + 3$ **31.** $y = \frac{5}{4}(x - 3)^2$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 17–25
Example 2: Exs. 17–19, 26–31
Example 3: Exs. 17–19, 32–37
Example 4: Exs. 38–49
Examples 5, 6: Exs. 51–54

FOCUS ON APPLICATIONS

TORQUE, the focus of Ex. 51, is the "twisting force" produced by the crankshaft in a car's engine. As torque increases, a car is able to accelerate more quickly.

APPLICATION LINK
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GRAPHING WITH INTERCEPT FORM **Graph the quadratic function. Label the vertex, axis of symmetry, and *x*-intercepts.**

32. $y = (x - 2)(x - 6)$ **33.** $y = 4(x + 1)(x - 1)$ **34.** $y = -(x + 3)(x + 5)$

35. $y = \frac{1}{3}(x + 4)(x + 1)$ **36.** $y = -\frac{1}{2}(x - 3)(x + 2)$ **37.** $y = -3x(x - 2)$

WRITING IN STANDARD FORM **Write the quadratic function in standard form.**

38. $y = (x + 5)(x + 2)$ **39.** $y = -(x + 3)(x - 4)$ **40.** $y = 2(x - 1)(x - 6)$

41. $y = -3(x - 7)(x + 4)$ **42.** $y = (5x + 8)(4x + 1)$ **43.** $y = (x + 3)^2 + 2$

44. $y = -(x - 5)^2 + 11$ **45.** $y = -6(x - 2)^2 - 9$ **46.** $y = 8(x + 7)^2 - 20$

47. $y = -(9x + 2)^2 + 4x$ **48.** $y = -\frac{7}{3}(x + 6)(x + 3)$ **49.** $y = \frac{1}{2}(8x - 1)^2 - \frac{3}{2}$

50. **VISUAL THINKING** In parts (a) and (b), use a graphing calculator to examine how b and c affect the graph of $y = ax^2 + bx + c$.

a. Graph $y = x^2 + c$ for $c = -2, -1, 0, 1$, and 2. Use the same viewing window for all the graphs. How do the graphs change as c increases?

b. Graph $y = x^2 + bx$ for $b = -2, -1, 0, 1$, and 2. Use the same viewing window for all the graphs. How do the graphs change as b increases?

51. **AUTOMOBILES** The engine torque y (in foot-pounds) of one model of car is given by

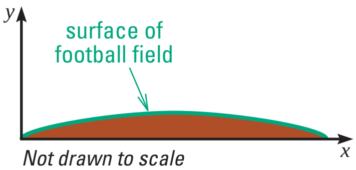
$$y = -3.75x^2 + 23.2x + 38.8$$

where x is the speed of the engine (in thousands of revolutions per minute). Find the engine speed that maximizes torque. What is the maximum torque?

52. **SPORTS** Although a football field appears to be flat, its surface is actually shaped like a parabola so that rain runs off to either side. The cross section of a field with synthetic turf can be modeled by

$$y = -0.000234(x - 80)^2 + 1.5$$

where x and y are measured in feet. What is the field's width? What is the maximum height of the field's surface? ▶ Source: Boston College



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53. **PHYSIOLOGY** Scientists determined that the rate y (in calories per minute) at which you use energy while walking can be modeled by

$$y = 0.00849(x - 90.2)^2 + 51.3, \quad 50 \le x \le 150$$

where x is your walking speed (in meters per minute). Graph the function on the given domain. Describe how energy use changes as walking speed increases. What speed minimizes energy use? ▶ Source: *Bioenergetics and Growth*

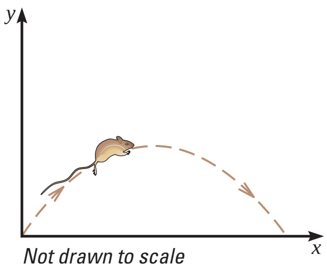
STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 54.

54. **BIOLOGY CONNECTION** The woodland jumping mouse can hop surprisingly long distances given its small size. A relatively long hop can be modeled by

$$y = -\frac{2}{9}x(x - 6)$$

where x and y are measured in feet. How far can a woodland jumping mouse hop? How high can it hop?
▶ Source: University of Michigan Museum of Zoology



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Test Preparation

55. MULTI-STEP PROBLEM A kernel of popcorn contains water that expands when the kernel is heated, causing it to pop. The equations below give the "popping volume" y (in cubic centimeters per gram) of popcorn with moisture content x (as a percent of the popcorn's weight). ▶ Source: *Cereal Chemistry*

Hot-air popping: $y = -0.761x^2 + 21.4x - 94.8$

Hot-oil popping: $y = -0.652x^2 + 17.7x - 76.0$

a. For hot-air popping, what moisture content maximizes popping volume? What is the maximum volume?

b. For hot-oil popping, what moisture content maximizes popping volume? What is the maximum volume?

c. The moisture content of popcorn typically ranges from 8% to 18%. Graph the equations for hot-air and hot-oil popping on the interval $8 \le x \le 18$.

d. *Writing* Based on the graphs from part (c), what general statement can you make about the volume of popcorn produced from hot-air popping versus hot-oil popping for any moisture content in the interval $8 \le x \le 18$?

★ **Challenge**

56. LOGICAL REASONING Write $y = a(x - h)^2 + k$ and $y = a(x - p)(x - q)$ in standard form. Knowing that the vertex of the graph of $y = ax^2 + bx + c$ occurs at $x = -\frac{b}{2a}$, show that the vertex for $y = a(x - h)^2 + k$ occurs at $x = h$ and that the vertex for $y = a(x - p)(x - q)$ occurs at $x = \frac{p + q}{2}$.

EXTRA CHALLENGE
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MIXED REVIEW

SOLVING LINEAR EQUATIONS **Solve the equation.** (Review 1.3 for 5.2)

57. $x - 2 = 0$ **58.** $2x + 5 = 0$ **59.** $-4x - 7 = 21$

60. $3x + 9 = -x + 1$ **61.** $6(x + 8) = 18$ **62.** $5(4x - 1) = 2(x + 3)$

63. $0.6x = 0.2x + 2.8$ **64.** $\frac{7x}{8} - \frac{3x}{5} = \frac{11}{2}$ **65.** $\frac{5x}{12} + \frac{1}{4} = \frac{x}{6} - \frac{1}{2}$

GRAPHING IN THREE DIMENSIONS **Sketch the graph of the equation. Label the points where the graph crosses the *x*-, *y*-, and *z*-axes.** (Review 3.5)

66. $x + y + z = 4$ **67.** $x + y + 2z = 6$ **68.** $3x + 4y + z = 12$

69. $5x + 5y + 2z = 10$ **70.** $2x + 7y + 3z = 42$ **71.** $x + 3y - 3z = 9$

USING CRAMER'S RULE **Use Cramer's rule to solve the linear system.** (Review 4.3)

72. $x + y = 1$
$-5x + y = 19$

73. $2x + y = 5$
$3x - 4y = 2$

74. $7x - 10y = -15$
$x + 2y = -9$

75. $5x + 2y + 2z = 4$
$3x + y - 6z = -4$
$-x - y - z = 1$

76. $x + 3y + z = 5$
$-x + y + z = 7$
$2x - 7y + 5z = 28$

77. $2x - 3y - 9z = 11$
$6x + y - z = 45$
$9x - 2y + 4z = 56$

78. **WEATHER** In January, 1996, rain and melting snow caused the depth of the Susquehanna River in Pennsylvania to rise from 7 feet to 22 feet in 14 hours. Find the average rate of change in the depth during that time. (Review 2.2)