4.3 Determinants and Cramer's Rule

What you should learn

GOAL 1 Evaluate determinants of 2 × 2 and 3 × 3 matrices.

GOAL 2 Use Cramer's rule to solve systems of linear equations, as applied in **Example 5**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the area of the Golden Triangle of India in **Ex. 58**.

REAL LIFE

GOAL 1 EVALUATING DETERMINANTS

Associated with each square matrix is a real number called its **determinant**. The determinant of a matrix A is denoted by det A or by $|A|$.

THE DETERMINANT OF A MATRIX

DETERMINANT OF A 2 × 2 MATRIX

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

The determinant of a 2 × 2 matrix is the difference of the products of the entries on the diagonals.

DETERMINANT OF A 3 × 3 MATRIX

1. **Repeat the first two columns to the right of the determinant.**
2. **Subtract the sum of the products in red from the sum of the products in blue.**

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = (aei + bfg + cdh) - (gec + hfa + idb)$$

EXAMPLE 1 *Evaluating Determinants*

Evaluate the determinant of the matrix.

a. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

SOLUTION

a. $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 1(5) - 2(3) = 5 - 6 = -1$

b. $\begin{vmatrix} 2 & -1 & 3 \\ -2 & 0 & 1 \\ 1 & 2 & 4 \end{vmatrix}\begin{matrix} 2 & -1 \\ -2 & 0 \\ 1 & 2 \end{matrix} = [0 + (-1) + (-12)] - (0 + 4 + 8) = -13 - 12$

$= -25$

. . . .

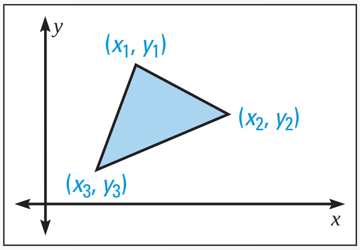
You can use a determinant to find the area of a triangle whose vertices are points in a coordinate plane.

AREA OF A TRIANGLE

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by

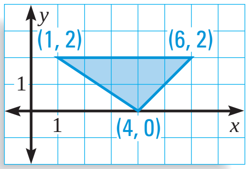
$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol $\pm$ indicates that the appropriate sign should be chosen to yield a positive value.



EXAMPLE 2 *The Area of a Triangle*

The area of the triangle shown is:

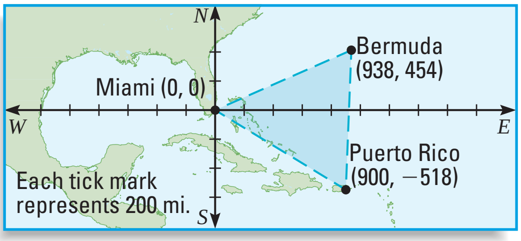


$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 4 & 0 & 1 \\ 6 & 2 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{2}[(0 + 12 + 8) - (0 + 2 + 8)] = 5$$

EXAMPLE 3 *The Area of a Triangular Region*

BERMUDA TRIANGLE The Bermuda Triangle is a large triangular region in the Atlantic Ocean. Many ships and airplanes have been lost in this region. The triangle is formed by imaginary lines connecting Bermuda, Puerto Rico, and Miami, Florida. Use a determinant to estimate the area of the Bermuda Triangle.



SOLUTION

The approximate coordinates of the Bermuda Triangle's three vertices are (938, 454), (900, −518), and (0, 0). So, the area of the region is as follows:

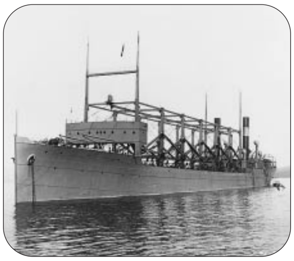
$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} 938 & 454 & 1 \\ 900 & -518 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{2}[(-485{,}884 + 0 + 0) - (0 + 0 + 408{,}600)]$$

$$= 447{,}242$$

▶ The area of the Bermuda Triangle is about 447,000 square miles.

FOCUS ON APPLICATIONS



REAL LIFE BERMUDA TRIANGLE
The U.S.S. *Cyclops*, shown above, disappeared in the Bermuda Triangle in March, 1918.

GOAL 2 USING CRAMER'S RULE

You can use determinants to solve a system of linear equations. The method, called **Cramer's rule** and named after the Swiss mathematician Gabriel Cramer (1704–1752), uses the **coefficient matrix** of the linear system.

LINEAR SYSTEM	COEFFICIENT MATRIX
$ax + by = e$ $cx + dy = f$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

CRAMER'S RULE FOR A 2 X 2 SYSTEM

Let A be the coefficient matrix of this linear system:

$$ax + by = e$$
$$cx + dy = f$$

If $\det A \neq 0$, then the system has exactly one solution. The solution is:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$$

In Cramer's rule, notice that the denominator for x and y is the determinant of the coefficient matrix of the system. The numerators for x and y are the determinants of the matrices formed by using the column of constants as replacements for the coefficients of x and y, respectively.

EXAMPLE 4 *Using Cramer's Rule for a 2 × 2 System*

Use Cramer's rule to solve this system: $\begin{aligned} 8x + 5y &= 2 \\ 2x - 4y &= -10 \end{aligned}$

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

SOLUTION

Evaluate the determinant of the coefficient matrix.

$$\begin{vmatrix} 8 & 5 \\ 2 & -4 \end{vmatrix} = -32 - 10 = -42$$

Apply Cramer's rule since the determinant is not 0.

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42} = \frac{-8 - (-50)}{-42} = \frac{42}{-42} = -1$$

$$y = \frac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42} = \frac{-80 - 4}{-42} = \frac{-84}{-42} = 2$$

▶ The solution is $(-1, 2)$.

✓**CHECK** Check this solution in the original equations.

$8(-1) + 5(2) \stackrel{?}{=} 2$ $\qquad 2(-1) - 4(2) \stackrel{?}{=} -10$

$2 = 2$ ✓ $\qquad -10 = -10$ ✓

CRAMER'S RULE FOR A 3 X 3 SYSTEM

Let A be the coefficient matrix of this linear system:

$$\begin{aligned} ax + by + cz &= j \\ dx + ey + fz &= k \\ gx + hy + iz &= l \end{aligned}$$

If det $A \neq 0$, then the system has exactly one solution. The solution is:

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\det A}, \quad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\det A}, \quad \text{and} \quad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\det A}$$

EXAMPLE 5 Using Cramer's Rule for a 3 × 3 System

SCIENCE CONNECTION The atomic weights of three compounds are shown. Use a linear system and Cramer's rule to find the atomic weights of carbon (C), hydrogen (H), and oxygen (O).

Compound	Formula	Atomic weight
Methane	CH_4	16
Glycerol	$C_3H_8O_3$	92
Water	H_2O	18

SOLUTION

Write a linear system using the formula for each compound. Let C, H, and O represent the atomic weights of carbon, hydrogen, and oxygen.

$$\begin{aligned} C + 4H &= 16 \\ 3C + 8H + 3O &= 92 \\ 2H + O &= 18 \end{aligned}$$

Evaluate the determinant of the coefficient matrix.

$$\begin{vmatrix} 1 & 4 & 0 \\ 3 & 8 & 3 \\ 0 & 2 & 1 \end{vmatrix} = (8 + 0 + 0) - (0 + 6 + 12) = -10$$

Apply Cramer's rule since the determinant is not 0.

$$C = \frac{\begin{vmatrix} 16 & 4 & 0 \\ 92 & 8 & 3 \\ 18 & 2 & 1 \end{vmatrix}}{-10} = \frac{-120}{-10} = 12$$ **Atomic weight of carbon**

$$H = \frac{\begin{vmatrix} 1 & 16 & 0 \\ 3 & 92 & 3 \\ 0 & 18 & 1 \end{vmatrix}}{-10} = \frac{-10}{-10} = 1$$ **Atomic weight of hydrogen**

$$O = \frac{\begin{vmatrix} 1 & 4 & 16 \\ 3 & 8 & 92 \\ 0 & 2 & 18 \end{vmatrix}}{-10} = \frac{-160}{-10} = 16$$ **Atomic weight of oxygen**

▶ The weights of carbon, hydrogen, and oxygen are 12, 1, and 16, respectively.

FOCUS ON CAREERS

REAL LIFE **CHEMIST** Chemists research and put to practical use knowledge about chemicals. Research on the chemistry of living things sparks advances in medicine, agriculture, and other fields.

INTERNET **CAREER LINK** www.mcdougallittell.com

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain Cramer's rule and how it is used.

Concept Check ✓

2. Can two different matrices have the same determinant? If so, give an example.

3. **ERROR ANALYSIS** Find the error in each calculation.

   a. $\begin{vmatrix} 3 & -1 \\ 5 & 2 \end{vmatrix} = 6 - 5 = 1$

   b. $\begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix} = -2 - 12 = -14$

4. To use Cramer's rule to solve a linear system, what must be true of the determinant of the coefficient matrix?

Skill Check ✓

Evaluate the determinant of the matrix.

5. $\begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix}$

6. $\begin{bmatrix} -1 & 4 \\ 5 & -1 \end{bmatrix}$

7. $\begin{bmatrix} 8 & -2 \\ -2 & 4 \end{bmatrix}$

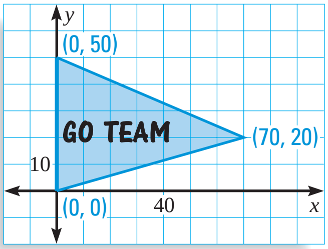
Use Cramer's rule to solve the linear system.

8. $6x - 8y = 4$
   $4x - 5y = -4$

9. $2x + 7y = -3$
   $3x - 8y = -23$

10. $12x - 2y = 2$
    $-14x + 11y = 51$

11. **SCHOOL SPIRIT** You are making a large pennant for your school football team. A diagram of the pennant is shown at the right. The coordinates given are measured in inches. How many square inches of material will you need to make the pennant?



PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 945.

2 × 2 DETERMINANTS **Evaluate the determinant of the matrix.**

12. $\begin{bmatrix} -4 & 2 \\ 5 & -2 \end{bmatrix}$

13. $\begin{bmatrix} 8 & 0 \\ -1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 9 & 3 \\ -2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} -7 & 11 \\ -7 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 4 & 0 \\ -3 & 4 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 8 \\ 5 & 9 \end{bmatrix}$

18. $\begin{bmatrix} -6 & 5 \\ -3 & 9 \end{bmatrix}$

19. $\begin{bmatrix} 0 & -3 \\ 8 & 10 \end{bmatrix}$

20. $\begin{bmatrix} 12 & 2 \\ -5 & 8 \end{bmatrix}$

3 × 3 DETERMINANTS **Evaluate the determinant of the matrix.**

21. $\begin{bmatrix} 12 & 4 & -1 \\ -2 & 3 & 2 \\ 5 & 8 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 5 & -9 & 4 \\ 4 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 5 & 2 \\ 10 & 13 & -4 \\ -5 & 4 & -1 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 16 & -2 \\ 20 & 4 & 2 \\ 7 & 1 & -4 \end{bmatrix}$

25. $\begin{bmatrix} -4 & 0 & -1 \\ 0 & 8 & 9 \\ 0 & 5 & 2 \end{bmatrix}$

26. $\begin{bmatrix} 8 & 2 & 9 \\ 12 & 3 & 9 \\ 3 & 13 & 4 \end{bmatrix}$

27. $\begin{bmatrix} 3 & 12 & -1 \\ 10 & 9 & 0 \\ -5 & 6 & -2 \end{bmatrix}$

28. $\begin{bmatrix} -3 & 2 & 20 \\ -10 & 9 & 18 \\ 11 & 15 & 12 \end{bmatrix}$

29. $\begin{bmatrix} 15 & 4 & -10 \\ -10 & 0 & 6 \\ -8 & 2 & -14 \end{bmatrix}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 12–29
Example 2: Exs. 30–35
Example 3: Exs. 54–58
Example 4: Exs. 36–44, 59
Example 5: Exs. 45–53, 60

AREA OF A TRIANGLE **Find the area of the triangle with the given vertices.**

30. $A(0, 1), B(2, 7), C(5, 5)$

31. $A(3, 6), B(3, 0), C(1, 3)$

32. $A(6, -1), B(2, 2), C(4, 8)$

33. $A(-4, 2), B(3, -1), C(-2, -2)$

34. $A(2, -6), B(-1, -4), C(0, 2)$

35. $A(1, 3), B(-2, 6), C(-1, 1)$

USING CRAMER'S RULE **Use Cramer's rule to solve the linear system.**

36. $2x + y = 3$
$5x + 6y = 4$

37. $7x - 5y = 11$
$3x + 10y = -56$

38. $9x + 2y = 7$
$4x - 3y = 42$

39. $x + 7y = -3$
$3x - 5y = 17$

40. $-x - 12y = 44$
$12x - 15y = -51$

41. $4x - 3y = 18$
$8x - 7y = 34$

42. $4x - 5y = 13$
$2x - 7y = 24$

43. $8x - 9y = 32$
$-5x + 7y = 40$

44. $3x + 10y = 50$
$12x + 15y = 64$

SOLVING SYSTEMS **Use Cramer's rule to solve the linear system.**

45. $x + 2y - 3z = -2$
$x - y + z = -1$
$3x + 4y - 4z = 4$

46. $x + 3y - z = 1$
$-2x - 6y + z = -3$
$3x + 5y - 2z = 4$

47. $3x + 2y - 5z = -10$
$6x - z = 8$
$-y + 3z = -2$

48. $x + 2y + z = 9$
$x + y + z = 3$
$5x - 2z = -1$

49. $4x + y + 6z = 7$
$3x + 3y + 2z = 17$
$-x - y + z = -9$

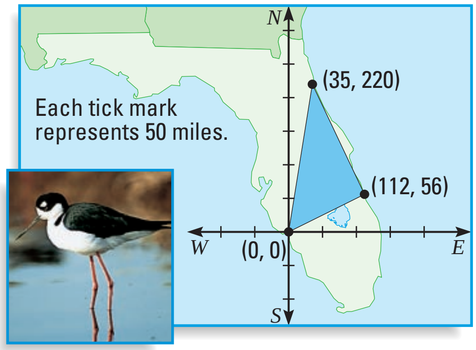
50. $x + 4y - z = -7$
$2x - y + 2z = 15$
$-3x + y - 3z = -22$

51. $2x + y + z = 5$
$x + 4y - 2z = 9$
$6x + 5y = 16$

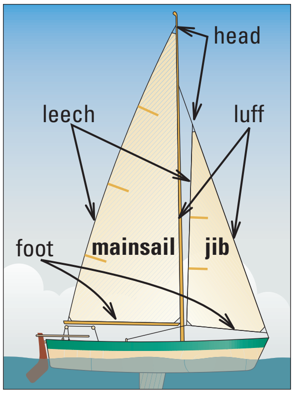
52. $-x + 2y + 7z = 13$
$2x - y - 2z = -2$
$3x + 5y + 2z = -14$

53. $-3x + y + 2z = -14$
$9x - y + 2z = -8$
$8x + 5y - 4z = 6$

54. **BIRDS** Black-necked stilts are birds that live throughout Florida and surrounding areas but breed mostly in the triangular region shown on the map. Estimate the area of this region. The coordinates given are measured in miles.

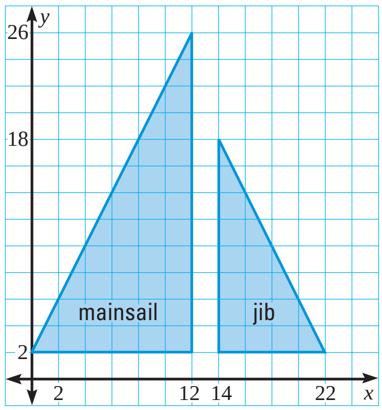


FOCUS ON APPLICATIONS



REAL LIFE **SAILING** The edges of a sail are called the luff, leech, and foot. The luff length of the jib is usually 80% to 90% of the distance from the deck to the head of the jib.

SAILING **In Exercises 55–57, use the following information.** On a Marconi-rigged sloop, there are two triangular sails, a mainsail and a jib. These sails are shown in a coordinate plane at the right. The coordinates in the plane are measured in feet.

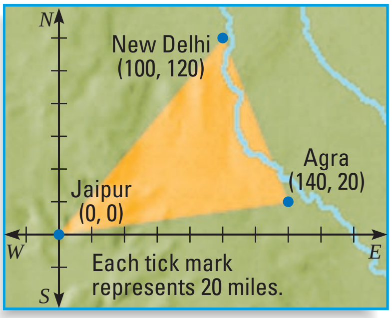


55. Find the area of the mainsail shown.

56. Find the area of the jib shown.

57. Suppose you are making a scale model of the sailboat with the sails shown using a scale of 1 in. = 6 ft. What is the area of the model's mainsail?

58. **SOCIAL STUDIES CONNECTION** The Golden Triangle refers to a large triangular region in India. The Taj Mahal is one of the many wonders that lie within the boundaries of this triangle. The triangle is formed by imaginary lines that connect the cities of New Delhi, Jaipur, and Agra. Use the coordinates on the map and a determinant to estimate the area of the Golden Triangle. The coordinates given are measured in miles.



59. **BUYING GASOLINE** You fill up your car with 10 gallons of premium gasoline and fill a small gas can with 2 gallons of regular gasoline for your lawn mower. You pay the cashier \$13.56. The price of premium gasoline is 12 cents more per gallon than the price of regular gasoline. Use a linear system and Cramer's rule to find the price per gallon for regular and premium gasoline.

60. **SCIENCE CONNECTION** The atomic weights of three compounds are shown.

Compound	Formula	Atomic weight
Tetrasulphur tetranitride	S_4N_4	184
Sulphur hexaflouride	SF_6	146
Dinitrogen tetraflouride	N_2F_4	104

Use a linear system and Cramer's rule to find the atomic weights of sulphur (S), nitrogen (N), and flourine (F).

61. **LOGICAL REASONING** Explain what happens to the determinant of a matrix when you switch two rows or two columns.

Test Preparation

QUANTITATIVE COMPARISON **In Exercises 62 and 63, choose the statement that is true about the given quantities.**

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
62.	The area of a triangle with vertices $(-3, 4)$, $(4, 2)$, and $(1, -2)$	The area of a triangle with vertices $(4, 2)$, $(1, -2)$, and $(3, -4)$
63.	$\det \begin{bmatrix} -5 & 6 \\ -2 & 10 \end{bmatrix}$	$\det \begin{bmatrix} -7 & 1 \\ 3 & 5 \end{bmatrix}$

★ Challenge

64. **DETERMINANT RELATIONSHIPS** Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$.

a. How is det AB related to det A and det B?

b. How is det kA related to det A if k is a constant? Check your answer using matrix B and several other 2×2 matrices.

EXTRA CHALLENGE
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MIXED REVIEW

EVALUATING FUNCTIONS **Find the indicated value of $f(x)$.** (Review 2.1)

65. $f(x) = x - 10, f(7)$

66. $f(x) = 3x + 7, f(-2)$

67. $f(x) = -x^2 + 5, f(-1)$

68. $f(x) = x^2 - 2x - 4, f(7)$

69. $f(x) = x^2 + 4x - 1, f\left(\frac{1}{2}\right)$

70. $f(x) = x^5 - 2x - 10, f(3)$

GRAPHING SYSTEMS **Graph the system of linear inequalities.** (Review 3.3)

71. $x < 3$
$x > -2$

72. $y \geq 2x - 3$
$y > -5x - 8$

73. $y > -x - 5$
$y > 3x + 1$

74. $x + y > 3$
$4x + y < 4$

75. $2x - y \geq 2$
$5x - y \geq 2$

76. $4x - 3y > 1$
$-x + y \geq 4$

MULTIPLYING MATRICES **Find the product.** (Review 4.2 for 4.4)

77. $\begin{bmatrix} -2 & -4 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 3 & -3 \end{bmatrix}$

78. $\begin{bmatrix} 7 & -1 \\ 4 & -10 \end{bmatrix}\begin{bmatrix} 0 & -3 \\ 4 & 8 \end{bmatrix}$

79. $\begin{bmatrix} 11 & -2 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -8 & 3 \\ 8 & -1 \end{bmatrix}$

80. $\begin{bmatrix} 3 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 10 & 9 \\ 12 & 16 \end{bmatrix}$

81. $\begin{bmatrix} 0.5 & 3 \\ 0.2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0.6 \\ 4 & 0.8 \end{bmatrix}$

82. $\begin{bmatrix} -2 & 1.3 \\ 1.5 & -3 \end{bmatrix}\begin{bmatrix} 1.6 & 6 \\ -4 & 1.9 \end{bmatrix}$

QUIZ 1

Self-Test for Lessons 4.1–4.3

Perform the indicated operation(s). (Lessons 4.1, 4.2)

1. $\begin{bmatrix} -2 & 5 & 10 \\ 4 & -6 & 8 \end{bmatrix} + \begin{bmatrix} -3 & -1 & 5 \\ -2 & -8 & -7 \end{bmatrix}$

2. $\begin{bmatrix} -8 & 0 \\ 5 & -2 \end{bmatrix} - \begin{bmatrix} -3 & 7 \\ 5 & -1 \end{bmatrix}$

3. $-2\begin{bmatrix} 7 & -2 \\ 4 & 9 \end{bmatrix} + 2\begin{bmatrix} 6 & -3 \\ -5 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 4 & -6 & 10 \\ 3 & 6 & 0 \\ 9 & -4 & 5 \end{bmatrix} - 4\begin{bmatrix} 2 & -1 & -3 \\ 0 & 6 & -5 \\ -2 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 8 & -1 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ -2 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -1 & 3 \\ 2 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 9 & -3 \\ 4 & -6 \end{bmatrix}$

Evaluate the determinant of the matrix. (Lesson 4.3)

7. $\begin{bmatrix} -4 & 3 \\ -6 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 9 & -3 \\ 6 & -2 \end{bmatrix}$

9. $\begin{bmatrix} -1 & 2 & 3 \\ 5 & 0 & -2 \\ 6 & 8 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 12 & 5 & -6 \\ 2 & 2 & 3 \\ 1 & 0 & -3 \end{bmatrix}$

Use Cramer's rule to solve the linear system. (Lesson 4.3)

11. $-8x + y = -6$
$-5x + 4y = 3$

12. $3x - 2y = 10$
$-6x + y = -7$

13. $5x + 4y = 12$
$3x - 6y = 3$

14. $4x + y + 6z = 2$
$2x + 2y + 4z = 1$
$-x - y + z = -5$

15. $x + y + 4z = 7$
$2x - 3y - z = -24$
$-4x + 2y + 2z = 8$

16. $3x + 3y - 2z = -18$
$-5x - 2y - 3z = -1$
$7x + y + 6z = 14$

17. **GARDENING** You are planning to turn a triangular region of your yard into a garden. The vertices of the triangle are (0, 0), (5, 2), and (3, 6) where the coordinates are measured in feet. Find the area of the triangular region. (Lesson 4.3)