3.5 Graphing Linear Equations in Three Variables

What you should learn

GOAL 1 Graph linear equations in three variables and evaluate linear functions of two variables.

GOAL 2 Use functions of two variables to model **real-life** situations, such as finding the cost of planting a lawn in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as finding how many times to air a radio commercial in **Ex. 53**.

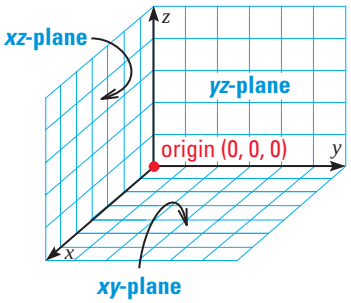
GOAL 1 GRAPHING IN THREE DIMENSIONS

Solutions of equations in three variables can be pictured with a **three-dimensional coordinate system**. To construct such a system, begin with the xy-coordinate plane in a horizontal position. Then draw the **z-axis** as a vertical line through the origin.

In much the same way that points in a two-dimensional coordinate system are represented by ordered pairs, each point in space can be represented by an **ordered triple** (x, y, z).

Drawing the point represented by an ordered triple is called *plotting* the point.

The three axes, taken two at a time, determine three coordinate planes that divide space into eight **octants**. The first octant is the one for which all three coordinates are positive.



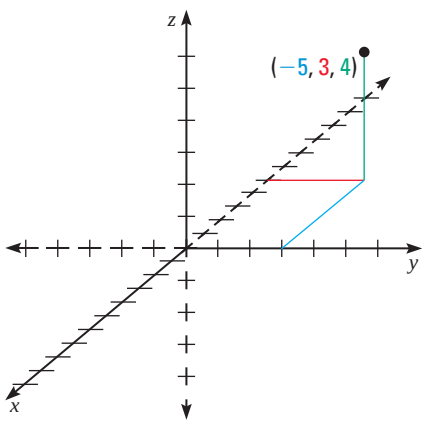
EXAMPLE 1 Plotting Points in Three Dimensions

Plot the ordered triple in a three-dimensional coordinate system.

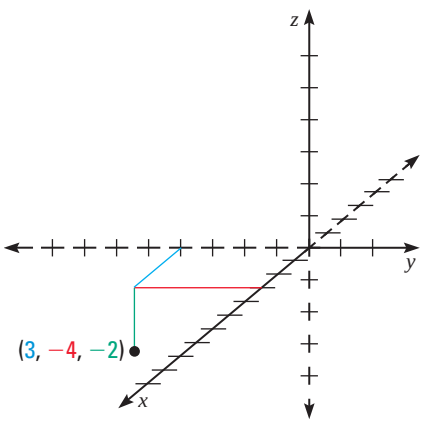
a. $(-5, 3, 4)$ **b.** $(3, -4, -2)$

SOLUTION

a. To plot $(-5, 3, 4)$, it helps to first find the point $(-5, 3)$ in the xy-plane. The point $(-5, 3, 4)$ lies four units above.



b. To plot $(3, -4, -2)$, find the point $(3, -4)$ in the xy-plane. The point $(3, -4, -2)$ lies two units below.



A **linear equation in three variables** x, y, and z is an equation of the form

$$ax + by + cz = d$$

where a, b, and c are not all zero. An ordered triple (x, y, z) is a *solution* of this equation if the equation is true when the values of x, y, and z are substituted into the equation. The *graph* of an equation in three variables is the graph of all its solutions. The graph of a linear equation in three variables is a plane.

EXAMPLE 2 Graphing a Linear Equation in Three Variables

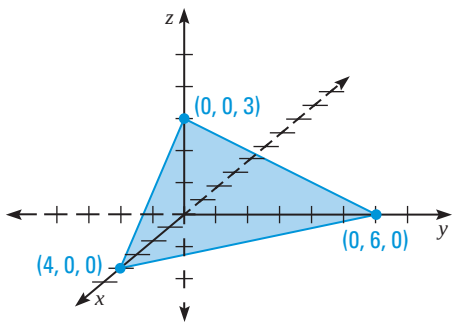
STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

Sketch the graph of $3x + 2y + 4z = 12$.

SOLUTION

Begin by finding the points at which the graph intersects the axes. Let $x = 0$ and $y = 0$, and solve for z to get $z = 3$. This tells you that the z-intercept is 3, so plot the point $(0, 0, 3)$. In a similar way, you can find that the x-intercept is 4 and the y-intercept is 6. After plotting $(0, 0, 3)$, $(4, 0, 0)$, and $(0, 6, 0)$, you can connect these points with lines to form the triangular region of the plane that lies in the first octant.



. . . . . .

A linear equation in x, y, and z can be written as a **function of two variables**. To do this, solve the equation for z. Then replace z with $f(x, y)$.

EXAMPLE 3 Evaluating a Function of Two Variables

a. Write the linear equation $3x + 2y + 4z = 12$ as a function of x and y.

b. Evaluate the function when $x = 1$ and $y = 3$. Interpret the result geometrically.

SOLUTION

a. $3x + 2y + 4z = 12$ — Write original equation.

$4z = 12 - 3x - 2y$ — Isolate z-term.

$z = \frac{1}{4}(12 - 3x - 2y)$ — Solve for z.

$f(x, y) = \frac{1}{4}(12 - 3x - 2y)$ — Replace z with $f(x, y)$.

STUDENT HELP

Study Tip
Remember that just as the notation $f(x)$ means the value of f at x, $f(x, y)$ means the value of f at the point (x, y).

b. $f(1, 3) = \frac{1}{4}(12 - 3(1) - 2(3)) = \frac{3}{4}$. This tells you that the graph of f contains the point $\left(1, 3, \frac{3}{4}\right)$.

FOCUS ON CAREERS

BIOTECHNICIAN There are about 40 grass species used as turf, two of which are discussed in Example 4. Turfgrass biotechnicians manipulate genetic traits of existing grass seed to breed improved varieties of grass.

CAREER LINK
www.mcdougallittell.com

GOAL 2 USING FUNCTIONS OF TWO VARIABLES IN REAL LIFE

EXAMPLE 4 *Modeling a Real-Life Situation*

LANDSCAPING You are planting a lawn and decide to use a mixture of two types of grass seed: bluegrass and rye. The bluegrass costs $2 per pound and the rye costs $1.50 per pound. To spread the seed you buy a spreader that costs $35.

a. Write a model for the total amount you will spend as a function of the number of pounds of bluegrass and rye.

b. Evaluate the model for several different amounts of bluegrass and rye, and organize your results in a table.

SOLUTION

a. Your total cost involves two variable costs (for the two types of seed) and one fixed cost (for the spreader).

VERBAL MODEL

Total cost = Bluegrass cost · Bluegrass amount + Rye cost · Rye amount + Spreader cost

LABELS

Total cost = C (dollars)

Bluegrass cost = **2** (dollars per pound)

Bluegrass amount = x (pounds)

Rye cost = **1.5** (dollars per pound)

Rye amount = y (pounds)

Spreader cost = **35** (dollars)

ALGEBRAIC MODEL

$$C = 2x + 1.5y + 35$$

b. To evaluate the function of two variables, substitute values of x and y into the function. For instance, when $x = 10$ and $y = 20$, the total cost is:

$C = 2x + 1.5y + 35$ Write original function.

$= 2(10) + 1.5(20) + 35$ Substitute for x and y.

$= 85$ Simplify.

The table shows the total cost for several different values of x and y.

Bluegrass (lb) \ Rye (lb)	0	10	20	30	40
10	$70	$85	$100	$115
20	$90	$105	$120	$135
30	$110	$125	$140	$155
40	$130	$145	$160	$175

STUDENT HELP

HOMEWORK HELP You can also use a spreadsheet to evaluate a function of two variables. For help with how to do this visit www.mcdougallittell.com

GUIDED PRACTICE

Vocabulary Check ✓

1. Write the general form of a linear equation in three variables. How is the solution of such an equation represented?

Concept Check ✓

2. **LOGICAL REASONING** Tell whether this statement is *true* or *false*: The graph of a linear equation in three variables consists of three different lines.

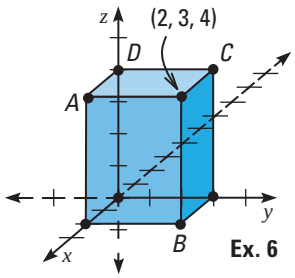
3. How are octants and quadrants similar?

4. Describe how you would graph a linear equation in three variables.

Skill Check ✓

5. Draw a three-dimensional coordinate system and plot the ordered triple $(2, -4, -6)$.

6. Write the coordinates of the vertices A, B, C, and D of the rectangular prism shown, given that one vertex is the point $(2, 3, 4)$.



Ex. 6

Sketch the graph of the equation. Label the points where the graph crosses the *x*-, *y*-, and *z*-axes.

7. $8x + 4y + 2z = 16$
8. $2x + 4y + 5z = 20$
9. $3x + 3y + 7z = 21$
10. $10x + 2y + 5z = 10$
11. $9x + 3y + 3z = 27$
12. $4x + y + 2z = 8$

Write the linear equation as a function of *x* and *y*. Then evaluate the function for the given values.

13. $6x + 6y + 3z = 9$, $f(1, 2)$
14. $-2x - y + z = 7$, $f(-3, 2)$
15. $8x + 2y + 4z = -16$, $f(5, 6)$
16. $5x - 10y - 5z = 15$, $f(2, 2)$

17. **TRAIL MIX** You are making bags of a trail mix called GORP (Good Old Raisins and Peanuts). The raisins cost \$2.25 per pound and the peanuts cost \$2.95 per pound. The package of bags for the trail mix costs \$2.65. Write a model for the total cost as a function of the number of pounds of raisins and peanuts you buy. Evaluate the model for 5 lb of raisins and 8 lb of peanuts.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 943.

PLOTTING POINTS **Plot the ordered triple in a three-dimensional coordinate system.**

18. $(2, 4, 0)$
19. $(4, -1, -6)$
20. $(5, -2, -2)$
21. $(0, 6, -3)$
22. $(3, 4, -2)$
23. $(-2, 1, 1)$
24. $(5, -1, 5)$
25. $(-3, 2, -7)$

SKETCHING GRAPHS **Sketch the graph of the equation. Label the points where the graph crosses the *x*-, *y*-, and *z*-axes.**

26. $x + y + z = 7$
27. $5x + 4y + 2z = 20$
28. $x + 6y + 4z = 12$
29. $12x + 3y + 8z = 24$
30. $2x + 18y + 3z = 36$
31. $7x + 9y + 21z = 63$
32. $7x + 7y + 2z = 14$
33. $6x + 4y + 3z = 10$
34. $3x + 5y + 3z = 15$
35. $\frac{1}{2}x + 4y - 3z = 8$
36. $5x + y + 2z = -4$
37. $-2x + 9y + 3z = 18$

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 18–25
Example 2: Exs. 26–37

continued on p. 174

STUDENT HELP

HOMEWORK HELP
continued from p. 173
Example 3: Exs. 38–45
Example 4: Exs. 48–52

EVALUATING FUNCTIONS **Write the linear equation as a function of *x* and *y*. Then evaluate the function for the given values.**

38. $6x + 2y + 3z = 18,\ f(2, 1)$

39. $-2x - 5y + 5z = 15,\ f\left(\frac{3}{2}, -2\right)$

40. $x + 6y + z = 10,\ f(-4, -1)$

41. $3x - \frac{3}{4}y + \frac{5}{2}z = 9,\ f(-3, 16)$

42. $-x - 2y - 7z = 14,\ f(-5, -10)$

43. $10x + 15y + 60z = 12,\ f\left(-3, \frac{4}{5}\right)$

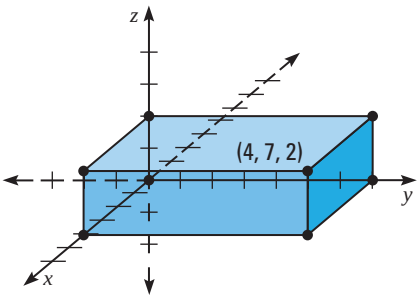
44. $x - 5y - z = 14,\ f(3, 6)$

45. $-x + 6y - 9z = 12,\ f\left(-\frac{1}{2}, 12\right)$

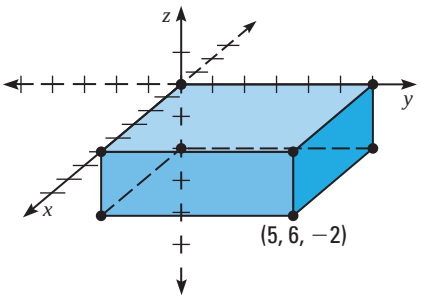
STUDENT HELP

Skills Review
For help with volume, see p. 914.

46. **GEOMETRY** **CONNECTION** Use the given point (4, 7, 2) to find the volume of the rectangular prism.



47. **GEOMETRY** **CONNECTION** Use the given point (5, 6, −2) to find the volume of the rectangular prism.



48. **HOME AQUARIUM** You want to buy an aquarium and stock it with goldfish and angelfish. The pet store sells goldfish for $.40 each and angelfish for $4 each. The aquarium starter kit costs $65. Write a model for the amount you will spend as a function of the number of goldfish and angelfish you buy. Make a table that shows the total cost for several different numbers of goldfish and angelfish.

49. **POTTERY** A craft store has paint-your-own pottery sessions available. You pick out a piece of pottery that ranges in price from $8 to $50 and pick out paint colors for $1.50 per color. The craft store charges a base fee of $16 for sitting time, brushes, glaze, and kiln time. Write a model for the total cost of making a piece of pottery as a function of the price of the pottery and the number of paint colors you use. Make a table that shows the total cost for several different pieces of pottery and numbers of paint colors.

50. **FLOWER ARRANGEMENT** You are buying tulips, carnations, and a glass vase to make a flower arrangement. The flower shop sells tulips for $.70 each and carnations for $.30 each. The glass vase costs $12. Write a model for the total cost of the flower arrangement as a function of the number of tulips and carnations you use. Make a table that shows the total cost for several different numbers of tulips and carnations.

51. **TRANSPORTATION** Every month you buy a local bus pass for $20 that is worth $.60 toward the fare for the local bus, the express bus, or the subway. The local bus costs $.60, the express bus costs $1.50, and the subway costs $.85. Write a model for the total cost of transportation in a month as a function of the number of times you take the express bus and the number of times you take the subway. Evaluate the model for 8 express bus rides and 10 subway rides. Make a table that shows the total cost for several different numbers of rides.

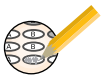
FOCUS ON APPLICATIONS

REAL LIFE **TRANSPORTATION**
The Massachusetts Bay Transit Authority (MBTA) is the nation's oldest subway system. On an average weekday, the MBTA serves about 1.2 million passengers on its bus, ferry, and train lines.

INTERNET **APPLICATION LINK**
www.mcdougallittell.com

52. **AFTER-SCHOOL JOBS** Several days after school you are a lifeguard at a community pool. On weekends you baby-sit to earn extra money. Lifeguarding pays $8 per hour and baby-sitting pays $6 per hour. You also get a weekly allowance of $10 for doing chores around the house. Write an equation for your total weekly earnings as a function of the number of hours you lifeguard and baby-sit. Make a table that shows several different amounts of weekly earnings.

Test Preparation



53. **MULTI-STEP PROBLEM** You are deciding how many times to air a 60 second commercial on a radio station. The station charges $100 for a 60 second spot during off-peak listening hours and $350 for a 60 second spot during peak listening hours. The company you have hired to make your commercial charges $500.

a. Write a model for the total amount that will be spent making and airing the commercial as a function of the number of times it is aired during off-peak and peak listening hours.

b. Evaluate the model for several different numbers of off-peak airings and peak airings. Organize your results in a table.

c. *Writing* Suppose your advertising budget is $4000. Using the table you made in part (b), can you air the commercial 8 times during off-peak hours and 8 times during peak hours? What combination of off-peak and peak airings would you recommend? Explain.

Challenge

WRITING EQUATIONS **Write an equation of the plane having the given *x*-, *y*-, and *z*-intercepts. Explain the method you used.**

EXTRA CHALLENGE
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54. x-intercept: 4
y-intercept: -2
z-intercept: 4

55. x-intercept: $\frac{3}{2}$
y-intercept: 12
z-intercept: 6

56. x-intercept: 4
y-intercept: -6
z-intercept: -9

MIXED REVIEW

SOLVING INEQUALITIES **Solve the inequality. Then graph the solution.** (Review 1.6)

57. $3 + x \leq 17$

58. $2x + 5 \geq 21$

59. $-x + 3 < 3x + 11$

60. $-13 < 6x - 1 < 11$

61. $24 \leq 2x - 12 \leq 30$

62. $-3 < 2x - 3 \leq 17$

TYPES OF LINES **Tell whether the lines are *parallel*, *perpendicular*, or *neither*.** (Review 2.2)

63. Line 1: through (1, 7) and (−3, −5)
Line 2: through (−6, 20) and (0, 2)

64. Line 1: through (4, −4) and (−16, 1)
Line 2: through (1, 5) and (5, 21)

65. Line 1: through (−2, 1) and (0, 3)
Line 2: through (2, 1) and (0, −1)

66. Line 1: through (0, 6) and (5, −2)
Line 2: through (−1, −1) and (7, 4)

67. **HOME CARPENTRY** You have budgeted $48.50 to purchase red oak and poplar boards to make a bookcase. Each red oak board costs $3.95 and each poplar board costs $3.10. You need a total of 14 boards for the bookcase. Write and solve a system of equations to find the number of red oak boards and the number of poplar boards you should buy. (Review 3.1, 3.2 for 3.6)