

3.4

Linear Programming

GOAL 1 USING LINEAR PROGRAMMING

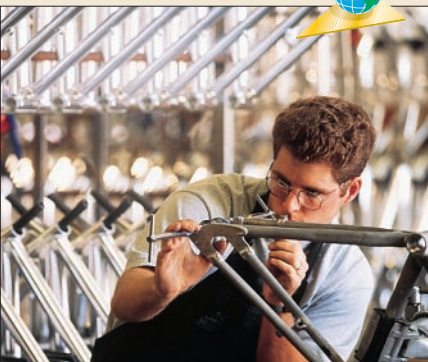
What you should learn

GOAL 1 Solve linear programming problems.

GOAL 2 Use linear programming to solve real-life problems, such as purchasing file cabinets so as to maximize storage capacity in Ex. 22.

Why you should learn it

▼ To solve real-life problems, such as how a bicycle manufacturer can maximize profit in Example 3.



Many real-life problems involve a process called **optimization**, which means finding the maximum or minimum value of some quantity. In this lesson you will study one type of optimization process called *linear programming*.

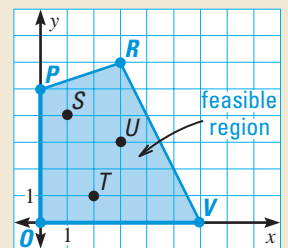
Linear programming is the process of optimizing a linear **objective function** subject to a system of linear inequalities called **constraints**. The graph of the system of constraints is called the **feasible region**.

ACTIVITY

Developing Concepts

Investigating Linear Programming

- 1 Evaluate the objective function $C = 2x + 4y$ for each labeled point in the feasible region at the right.
- 2 At which labeled point does the maximum value of C occur? At which labeled point does the minimum value of C occur?
- 3 What are the maximum and minimum values of C on the entire feasible region? Try other points in the region to see if you can find values of C that are greater or lesser than those you found in Step 2.

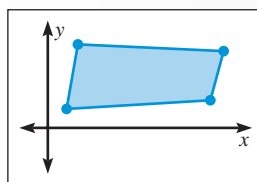


Constraints:
 $x \geq 0$
 $y \geq 0$
 $-x + 3y \leq 15$
 $2x + y \leq 12$

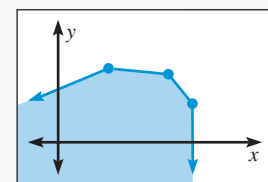
In the activity you may have discovered that the optimal values of the objective function occurred at vertices of the feasible region.

OPTIMAL SOLUTION OF A LINEAR PROGRAMMING PROBLEM

If an objective function has a maximum or a minimum value, then it must occur at a vertex of the feasible region. Moreover, the objective function will have both a maximum and a minimum value if the feasible region is bounded.



Bounded region



Unbounded region

EXAMPLE 1 Solving a Linear Programming Problem

Find the minimum value and the maximum value of

$$C = 3x + 4y \quad \text{Objective function}$$

subject to the following constraints.

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 8 \end{aligned} \quad \text{Constraints}$$

SOLUTION

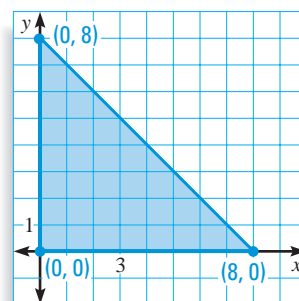
The feasible region determined by the constraints is shown. The three vertices are $(0, 0)$, $(8, 0)$, and $(0, 8)$. To find the minimum and maximum values of C , evaluate $C = 3x + 4y$ at each of the three vertices.

$$\text{At } (0, 0): C = 3(0) + 4(0) = 0 \quad \leftarrow \text{Minimum}$$

$$\text{At } (8, 0): C = 3(8) + 4(0) = 24$$

$$\text{At } (0, 8): C = 3(0) + 4(8) = 32 \quad \leftarrow \text{Maximum}$$

The minimum value of C is 0. It occurs when $x = 0$ and $y = 0$. The maximum value of C is 32. It occurs when $x = 0$ and $y = 8$.



EXAMPLE 2 A Region that is Unbounded

Find the minimum value and the maximum value of

$$C = 5x + 6y \quad \text{Objective function}$$

subject to the following constraints.

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\geq 5 \\ 3x + 4y &\geq 18 \end{aligned} \quad \text{Constraints}$$

SOLUTION

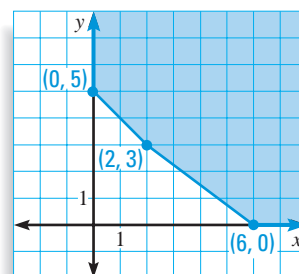
The feasible region determined by the constraints is shown. The three vertices are $(0, 5)$, $(2, 3)$, and $(6, 0)$. First evaluate $C = 5x + 6y$ at each of the vertices.

$$\text{At } (0, 5): C = 5(0) + 6(5) = 30$$

$$\text{At } (2, 3): C = 5(2) + 6(3) = 28$$

$$\text{At } (6, 0): C = 5(6) + 6(0) = 30$$

If you evaluate several other points in the feasible region, you will see that as the points get farther from the origin, the value of the objective function increases without bound. Therefore, the objective function has no maximum value. Since the value of the objective function is always at least 28, the minimum value is 28.



STUDENT HELP

Study Tip

You can find the coordinates of each vertex in the feasible region by solving systems of two linear equations. In Example 2 the vertex $(2, 3)$ is the solution of this system:

$$\begin{aligned} x + y &= 5 \\ 3x + 4y &= 18 \end{aligned}$$



BICYCLES In China bicycles are a popular means of transportation. In 1999 China had an estimated 700–800 million bicycles.

GOAL 2 LINEAR PROGRAMMING IN REAL LIFE

EXAMPLE 3 Using Linear Programming to Find the Maximum Profit

BICYCLE MANUFACTURING Two manufacturing plants make the same kind of bicycle. The table gives the hours of general labor, machine time, and technical labor required to make one bicycle in each plant. For the two plants combined, the manufacturer can afford to use up to 4000 hours of general labor, up to 1500 hours of machine time, and up to 2300 hours of technical labor per week. Plant A earns a profit of \$60 per bicycle and Plant B earns a profit of \$50 per bicycle. How many bicycles per week should the manufacturer make in each plant to maximize profit?

Resource	Hours per bicycle in Plant A	Hours per bicycle in Plant B
General labor	10	1
Machine time	1	3
Technical labor	5	2

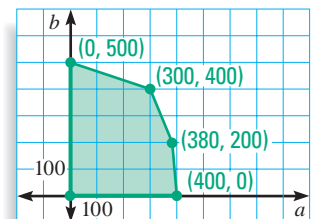
SOLUTION

Write an objective function. Let a and b represent the number of bicycles made in Plant A and Plant B, respectively. Because the manufacturer wants to maximize the profit P , the objective function is:

$$P = 60a + 50b$$

Write the constraints in terms of a and b . The constraints are given below and the feasible region determined by the constraints is shown at the right.

- $10a + b \leq 4000$ **General labor: up to 4000 hours**
- $a + 3b \leq 1500$ **Machine time: up to 1500 hours**
- $5a + 2b \leq 2300$ **Technical labor: up to 2300 hours**
- $a \geq 0$ **Cannot produce a negative amount**
- $b \geq 0$ **Cannot produce a negative amount**



Calculate the profit at each vertex of the feasible region.

- At $(0, 500)$: $P = 60(0) + 50(500) = 25,000$
- At $(300, 400)$: $P = 60(300) + 50(400) = 38,000$ ← **Maximum**
- At $(380, 200)$: $P = 60(380) + 50(200) = 32,800$
- At $(400, 0)$: $P = 60(400) + 50(0) = 24,000$
- At $(0, 0)$: $P = 60(0) + 50(0) = 0$

► The maximum profit is obtained by making 300 bicycles in Plant A and 400 bicycles in Plant B.

GUIDED PRACTICE

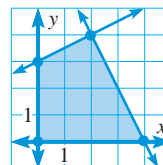
Vocabulary Check ✓

Concept Check ✓

1. Define linear programming.
2. How is the objective function used in a linear programming problem? How is the system of constraints used?
3. In a linear programming problem, which ordered pairs should be tested to find a minimum or maximum value?

Skill Check ✓

In Exercises 4 and 5, use the feasible region at the right.



4. What are the vertices of the feasible region?
5. What are the minimum and maximum values of the objective function $C = 5x + 7y$?

Find the minimum and maximum values of the objective function subject to the given constraints.

6. Objective function: $C = x + y$; Constraints: $y \leq 5, y \geq 0, y - 2x \geq 0$
7. Objective function: $C = 2x - y$; Constraints: $x \geq 0, x + y \leq 20, y \geq 3$
8. **PLANNING A FUNDRAISER** Your club plans to raise money by selling two sizes of fruit baskets. The plan is to buy small baskets for \$10 and sell them for \$16 and to buy large baskets for \$15 and sell them for \$25. The club president estimates that you will not sell more than 100 baskets. Your club can afford to spend up to \$1200 to buy the baskets. Find the number of small and large fruit baskets you should buy in order to maximize profit.

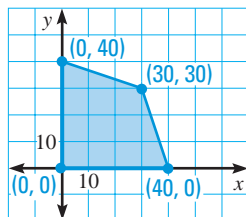
PRACTICE AND APPLICATIONS

STUDENT HELP

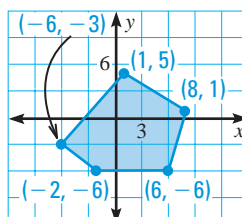
Extra Practice to help you master skills is on p. 943.

CHECKING VERTICES Find the minimum and maximum values of the objective function for the given feasible region.

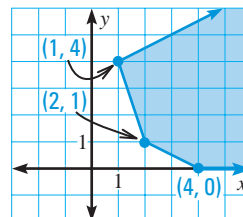
9. $C = x - y$



10. $C = 2x + 5y$



11. $C = 4x + 2y$



FINDING VALUES In Exercises 12–20, find the minimum and maximum values of the objective function subject to the given constraints.

12. Objective function:
 $C = 2x + 3y$

Constraints:
 $x \geq 0$
 $y \geq 0$
 $x + y \leq 9$

13. Objective function:
 $C = x + 4y$

Constraints:
 $x \geq 2$
 $x \leq 5$
 $y \geq 1$
 $y \leq 6$

14. Objective function:
 $C = 2x + y$

Constraints:
 $x \geq -5$
 $x \leq 0$
 $y \geq -2$
 $y \leq 2$

STUDENT HELP

HOMEWORK HELP

Examples 1, 2: Exs. 9–20
Example 3: Exs. 21–24

FOCUS ON PEOPLE



TOM FIRST AND TOM SCOTT

started a business in 1990, mixing juice in a blender and selling it from their boat. Like the problem described in Ex. 21, they had to figure out the best way of using available resources so as to maximize profit.

15. Objective function:

$$C = 10x + 7y$$

Constraints:

$$\begin{aligned} 0 \leq x &\leq 60 \\ 0 \leq y &\leq 45 \\ 5x + 6y &\leq 420 \end{aligned}$$

16. Objective function:

$$C = -2x + y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\geq 7 \\ 5x + 2y &\geq 20 \end{aligned}$$

17. Objective function:

$$C = 4x + 6y$$

Constraints:

$$\begin{aligned} -x + y &\leq 11 \\ x + y &\leq 27 \\ 2x + 5y &\leq 90 \end{aligned}$$

18. Objective function:

$$C = 5x + 4y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ y &\leq 8 \\ x + y &\leq 14 \\ 5x + y &\leq 50 \end{aligned}$$

19. Objective function:

$$C = 4x + 3y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ 2x + 3y &\geq 6 \\ 3x - 2y &\leq 9 \\ x + 5y &\leq 20 \end{aligned}$$

20. Objective function:

$$C = 10x + 3y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\geq 0 \\ 2x + y &\geq 4 \\ 2x + y &\leq 13 \end{aligned}$$

21. JUICE BLENDS A juice company makes two kinds of juice: Orangeade and Berry-fruity. One gallon of Orangeade is made by mixing 2.5 quarts of orange juice and 1.5 quarts of raspberry juice, while one gallon of Berry-fruity is made by mixing 3 quarts of raspberry juice and 1 quart of orange juice. A profit of \$.50 is made on every gallon of Orangeade sold, and a profit of \$.40 is made on every gallon of Berry-fruity sold. If the company has 150 gallons of raspberry juice and 125 gallons of orange juice on hand, how many gallons of each type of juice should be made to maximize profit?

22. FILE CABINETS An office manager is purchasing file cabinets and wants to maximize storage space. The office has 60 square feet of floor space for the cabinets and \$600 in the budget to purchase them. Cabinet A requires 3 square feet of floor space, has a storage capacity of 12 cubic feet, and costs \$75. Cabinet B requires 6 square feet of floor space, has a storage capacity of 18 cubic feet, and costs \$50. How many of each cabinet should the office manager buy?

23. HOME CANNING You have 180 tomatoes and 15 onions left over from your garden. You want to use these to make jars of tomato sauce and jars of salsa to sell at a farm stand. A jar of tomato sauce requires 10 tomatoes and 1 onion, and a jar of salsa requires 5 tomatoes and $\frac{1}{4}$ onion. You'll make a profit of \$2 on every jar of tomato sauce sold and a profit of \$1.50 on every jar of salsa sold. The farm stand wants at least three times as many jars of tomato sauce as jars of salsa. How many jars of each should you make to maximize profit?

24. NUTRITION You are planning a dinner of pinto beans and brown rice. You want to consume at least 2100 Calories and 44 grams of protein per day, but no more than 2400 milligrams of sodium and 73 grams of fat. So far today, you have consumed 1600 Calories, 24 grams of protein, 2370 milligrams of sodium, and 65 grams of fat. Pinto beans cost \$.57 per cup and brown rice costs \$.78 per cup. How many cups of pinto beans and brown rice should you make to minimize cost while satisfying your nutritional requirements?

Contents	1 cup pinto beans	1 cup brown rice (with salt)
Calories	265	230
Protein (g)	15	5
Sodium (mg)	3	10
Fat (g)	1	1

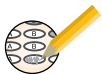
STUDENT HELP



HOMEWORK HELP

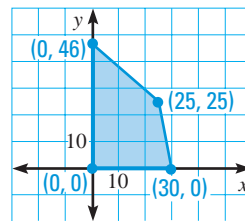
Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 24.

Test Preparation



25. **MULTIPLE CHOICE** Given the feasible region shown, what is the maximum value of the objective function $C = 2x + 6y$?

- (A) 0 (B) 60 (C) 200
(D) 276 (E) 326



26. **MULTIPLE CHOICE** Given the constraints $y \geq 0$, $y \leq x + 8$, and $y \geq 2x + 8$, what is the minimum value of the objective function $C = -2x - y$?

- (A) -8 (B) 16 (C) -16 (D) 8

★ Challenge

27. **CONSECUTIVE VERTICES** Find the value of the objective function at each vertex of the feasible region and at two points on each line segment connecting two vertices. What can you conclude?

a. **Objective function:**

$$C = 2x + 2y$$

Constraints:

$$y \leq 4$$

$$x \leq 5$$

$$x + y \leq 6$$

b. **Objective function:**

$$C = 5x - y$$

Constraints:

$$y \geq -1$$

$$x \leq 3$$

$$-5x + y \leq 4$$

EXTRA CHALLENGE

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MIXED REVIEW

GRAPHING EQUATIONS Graph the equation. Label any intercepts. (Review 2.3 for 3.5)

28. $x - y = 10$

29. $3x + 4y = -12$

30. $y = -3x + 2$

31. $5x - 15y = 15$

32. $y = -\frac{3}{4}x + 2$

33. $y = -\frac{1}{2}x + 7$

EVALUATING FUNCTIONS Evaluate the function for the given value of x . (Review 2.7)

$$f(x) = \begin{cases} 3x - 1, & \text{if } x < -2 \\ x - 5, & \text{if } x \geq -2 \end{cases} \quad g(x) = \begin{cases} -7x, & \text{if } x \leq 0 \\ 2x + 1, & \text{if } x > 0 \end{cases}$$

34. $f(0)$

35. $f(-2)$

36. $f(-10)$

37. $f(-1)$

38. $g(1)$

39. $g(-5)$

40. $g(-1)$

41. $g(7)$

GRAPHING SYSTEMS OF INEQUALITIES Graph the system of linear inequalities. (Review 3.3)

42. $x > 2$
 $y < 6$

43. $x + y \leq 5$
 $y > 0$

44. $x < -1$
 $x - y \geq 4$

45. $y < 5$
 $x \geq -1$
 $y \geq 1$

46. $-x + y > 2$
 $y > 0$
 $2x + y \leq 3$

47. $x + y \leq 6$
 $-\frac{1}{2}x + y \leq 3$
 $y \leq 3$

48. **AMUSEMENT CENTER** You have 30 tokens for playing video games and pinball. It costs 3 tokens to play a video game and 2 tokens to play pinball. You want to play an equal number of video games and pinball games. Use an algebraic model to find how many games of each you can play. (Review 1.5)

QUIZ 2

Self-Test for Lessons 3.3 and 3.4

Graph the system of linear inequalities. (Lesson 3.3)

$$\begin{aligned} 1. \quad & y > -2 \\ & x \geq -4 \\ & y \leq -x + 1 \end{aligned}$$

$$\begin{aligned} 2. \quad & y > -5 \\ & x \leq 2 \\ & y \leq x + 2 \end{aligned}$$

$$\begin{aligned} 3. \quad & x \leq 3 \\ & y < 2 \\ & y > -x + 1 \end{aligned}$$

Find the minimum and maximum values of the objective function $C = 5x + 2y$ subject to the given constraints. (Lesson 3.4)

4. Constraints:


$$\begin{aligned} x &\leq -2 \\ x &\geq -4 \\ y &\geq 1 \\ y &\leq 6 \end{aligned}$$

5. Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 2 \\ 2x + y &\leq 10 \\ x - 3y &\geq -3 \end{aligned}$$


6. Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ y &\leq 8 \\ x + y &\leq 14 \end{aligned}$$

7.  **MAXIMUM INCOME** You are stenciling wooden boxes to sell at a fair. It takes you 2 hours to stencil a small box and 3 hours to stencil a large box. You make a profit of \$10 for a small box and \$20 for a large box. If you have no more than 30 hours available to stencil and want at least 12 boxes to sell, how many of each size box should you stencil to maximize your profit? (Lesson 3.4)

MATH & History

Linear Programming in World War II

 **APPLICATION LINK**
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THEN

DURING WORLD WAR II, the need for efficient transportation of supplies inspired mathematician George Dantzig to develop linear programming.

The LST was a ship used during World War II that carried 3 ton trucks and 25 ton tanks. The upper deck could carry 27 trucks, but no tanks. The tank deck could carry 500 tons, but no more than 33 trucks.

1. What is the maximum number of tanks that an LST could hold?
2. What is the maximum number of trucks that an LST could hold?
3. Suppose an LST was to be loaded with as many tanks and trucks as possible, and at least three times as many trucks as tanks. What is the maximum number of tanks and trucks that could be loaded?



U.S. Marines loading military supplies on an LST (Landing Ship, Tank)

NOW

IN 1984 mathematician Narendra Karmarkar developed a new time-saving linear programming method. Today his method is used by industries that deal with allocation of resources, such as telephone companies, airlines, and manufacturers.



1947

George Dantzig develops simplex method.

L.V. Kantorovich and T.C. Koopmans receive Nobel Prize for their linear programming work.

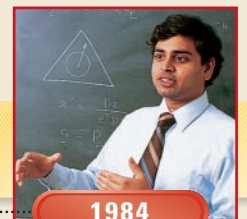


1975



1979

L.G. Khachyan develops ellipsoid method.



1984

N. Karmarkar devises a polynomial-time algorithm.