Normal Distributions

GOAL 1 **USING NORMAL DISTRIBUTIONS**
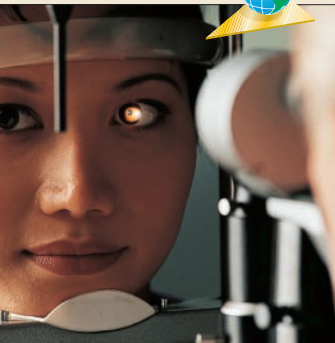
As statisticians began to study binomial distributions consisting of *n* trials with
probability *P* of success on each trial, they discovered that when *np* and $n(1 - p)$
are both greater than or equal to five, the distributions all resemble one another.
Here are two examples.
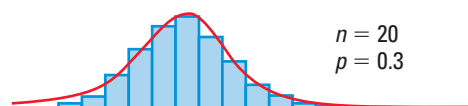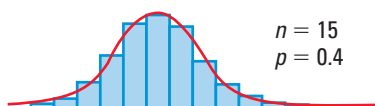
In both cases the binomial distribution can be approximated by a smooth,
symmetrical, bell-shaped curve called a **normal curve**. Areas under this curve
represent probabilities from **normal distributions**.

**CONCEPT
SUMMARY**   **AREAS UNDER A NORMAL CURVE**

The mean $\overline{x}$ and standard deviation σ of a normal distribution determine the
following areas.

- The total area under the curve is 1.
- 68% of the area lies within 1 standard deviation of the mean.
- 95% of the area lies within 2 standard deviations of the mean.
- 99.7% of the area lies within 3 standard deviations of the mean.

From the second bulleted statement above
and the symmetry of a normal curve, you
can deduce that 34% of the area lies within
1 standard deviation to the left of the mean,
and 34% of the area lies within 1 standard
deviation to the right of the mean. The
diagram shows other partial areas
(expressed as decimals rather than percents)
based on the properties of a normal curve.
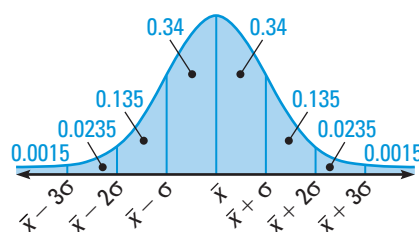
You can interpret these areas as probabilities.
In a normal distribution, the probability that
a randomly chosen *x*-value is between *a* and
b is given by the area under the normal curve
between *a* and *b*. For instance, the probability
that a randomly selected *x*-value is between 1
and 2 standard deviations to the right of the
mean is:

$$P(\overline{x} + \sigma \leq x \leq \overline{x} + 2\sigma) = 0.135$$
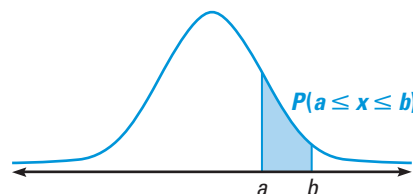
Many real-life distributions are normal or approximately normal.

Shopping

EXAMPLE 1 *Using a Normal Distribution*

A survey shows that the time spent by shoppers in supermarkets is normally distributed with a mean of 45 minutes and a standard deviation of 12 minutes.
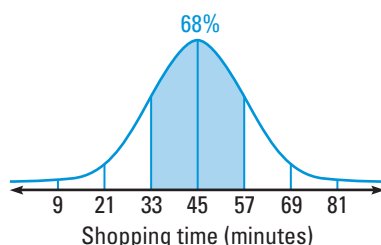
a. What percent of the shoppers at a supermarket will spend between 33 and 57 minutes in the supermarket?

b. What is the probability that a randomly chosen shopper will spend between 45 and 69 minutes in the supermarket?

SOLUTION

a. The given times of 33 minutes and 57 minutes represent one standard deviation on either side of the mean, as shown below. So, 68% of the shoppers will spend between 33 and 57 minutes in the supermarket.

68%

9 21 33 45 57 69 81
Shopping time (minutes)

b. The time of 45 minutes is the mean and the time of 69 minutes is two standard deviations to the right of the mean, as shown below. So, the probability that a randomly chosen shopper will spend between 45 and 69 minutes in the supermarket is 0.475.

$P(45 \le x \le 69) = 0.34 + 0.135 = 0.475$

9 21 33 45 57 69 81
Shopping time (minutes)
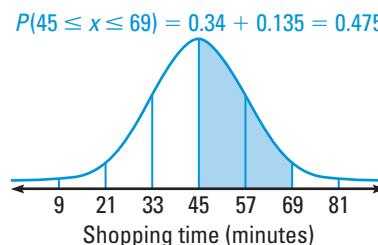
Health Statistics

EXAMPLE 2 *Using a Normal Distribution (Compound Event)*

According to a survey by the National Center for Health Statistics, the heights of adult men in the United States are normally distributed with a mean of 69 inches and a standard deviation of 2.75 inches. If you randomly choose 3 adult men, what is the probability that all three are 71.75 inches or taller?

SOLUTION

A height of 71.75 inches is one standard deviation to the right of the mean, as shown. The probability of randomly selecting a man who is this height or taller is:

$$P(x \ge 71.75) = 0.135 + 0.0235 + 0.0015$$

$$= 0.16$$

0.16

60.75 63.5 66.25 69 71.75 74.5 77.25
Height (inches)

Randomly choosing men are independent events, so the probability that all three randomly chosen men are 71.75 inches or taller is:

$$P(\text{all are 71.75 inches or taller}) = (0.16)^3$$

$$\approx 0.00410$$
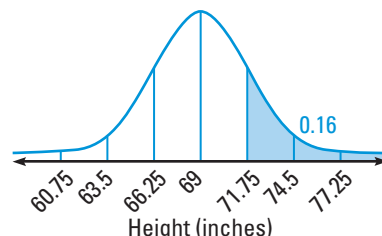
GOAL 2 APPROXIMATING BINOMIAL DISTRIBUTIONS

When n is large it can be tedious to compute binomial probabilities using the formula $P(k) = {}_nC_kp^k(1 - p)^{n-k}$. In such cases you may be able to use a normal distribution to approximate a binomial distribution.

NORMAL APPROXIMATION OF A BINOMIAL DISTRIBUTION

Consider the binomial distribution consisting of n trials with probability p of success on each trial. If $np \geq 5$ and $n(1 - p) \geq 5$, then the binomial distribution can be approximated by a normal distribution with a mean of

$$\overline{x} = np$$

and a standard deviation of

$$\sigma = \sqrt{np(1 - p)}.$$

EXAMPLE 3 *Finding a Binomial Probability*

SURVEYS According to a survey conducted by the Harris Poll, 29% of adults in the United States say that they or someone in their family plays soccer regularly. You are conducting a random survey of 238 adults. What is the probability that you will find at most 55 adults who come from a family in which someone plays soccer regularly?

SOLUTION

To answer the question using the binomial probability formula, you would have to calculate the following:

$$P(x \leq 55) = P(0) + P(1) + P(2) + \cdots + P(55)$$

This would be tedious. Instead you can approximate the answer with a normal distribution having a mean of

$$\overline{x} = np = 238(0.29) \approx 69$$

and a standard deviation of

$$\sigma = \sqrt{np(1 - p)} = \sqrt{238(0.29)(0.71)} \approx 7.$$

For this normal distribution, 55 is two standard deviations to the left of the mean. So, the probability that you will find at most 55 people from families in which someone plays soccer regularly is:

$$P(x \leq 55) \approx 0.0015 + 0.0235$$

$$= 0.025$$

People from "soccer families"

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If you had instead used the binomial probability formula in Example 3, you would have found the actual binomial probability to be approximately 0.0249. So, you can see that the normal approximation is a very good approximation.
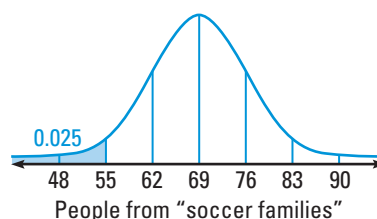
GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: A(n) _?_ can be used to approximate a binomial distribution when np and $n(1 - p)$ are both greater than or equal to 5.

Concept Check ✓

2. A normal curve is symmetric about what x-value?

3. What percent of the area under a normal curve lies within 1 standard deviation of the mean? within 2 standard deviations of the mean? within 3 standard deviations of the mean?

Skill Check ✓

A normal distribution has a mean of 10 and a standard deviation of 1. Find the probability that a randomly selected x-value is in the given interval.

4. between 8 and 12   5. between 7 and 13   6. between 8 and 11

7. at most 10   8. at least 12   9. at most 9

Find the mean and standard deviation of a normal distribution that approximates a binomial distribution consisting of n trials with probability p of success on each trial.

10. $n = 10, p = 0.5$ 11. $n = 17, p = 0.3$ 12. $n = 28, p = 0.2$

13. $n = 20, p = 0.25$ 14. $n = 12, p = 0.42$ 15. $n = 30, p = 0.17$

🌎 **COLORBLINDNESS In Exercises 16–18, use the fact that approximately 2% of people are colorblind, and consider a class of 460 students.**

16. What is the probability that 15 or fewer students are colorblind?

17. What is the probability that 12 or more students are colorblind?

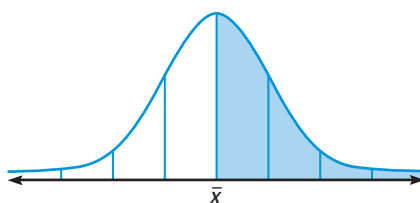18. What is the probability that between 6 and 18 students are colorblind?

PRACTICE AND APPLICATIONS

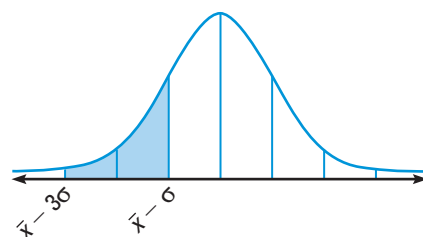**USING A NORMAL CURVE** Give the percent of the area under a normal curve represented by the shaded region.

19.

20.

21.
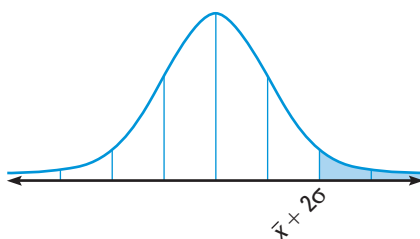
22.
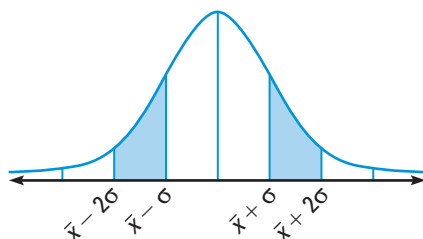
NORMAL DISTRIBUTIONS A normal distribution has a mean of 22 and a standard deviation of 3. Find the probability that a randomly selected x-value is in the given interval.

23. between 19 and 25   **24.** between 13 and 22   **25.** between 16 and 31

26. at most 25   **27.** at least 19   **28.** at most 28

FINDING PROBABILITIES A normal distribution has a mean of 64 and a standard deviation of 7. Find the given probability.

29. three randomly selected x-values are all 71 or greater

30. four randomly selected x-values are all 50 or less

31. two randomly selected x-values are both between 57 and 78

APPROXIMATING BINOMIAL DISTRIBUTIONS Find the mean and standard deviation of a normal distribution that approximates a binomial distribution consisting of n trials with probability p of success on each trial.

32. $n = 18, p = 0.7$ **33.** $n = 50, p = 0.1$ **34.** $n = 32, p = 0.8$

35. $n = 49, p = 0.12$ **36.** $n = 24, p = 0.67$ **37.** $n = 140, p = 0.06$

🌐 **DRIVE-THROUGH** In Exercises 38–40, use the following information.
A certain bank is busiest during the Friday evening rush hours from 3:00 P.M. until 6:00 P.M. During these hours the waiting time for drive-through customers is normally distributed with a mean of 8 minutes and a standard deviation of 2 minutes.

38. What percent of drive-through customers will wait for 10 minutes or longer during the Friday evening rush hours?

39. What is the probability that a customer will wait between 4 and 12 minutes during the Friday evening rush hours?

40. What is the probability that a customer will wait 2 minutes or less during the Friday evening rush hours?

🌐 **LIGHT BULBS** In Exercises 41–43, use the following information.
A company produces light bulbs having a life expectancy that is normally distributed with a mean of 1000 hours and a standard deviation of 50 hours.

41. What percent of the bulbs will last for 1000 hours or more?

42. What is the probability that a randomly chosen bulb will burn out in 900 hours or less?

43. What is the probability that a randomly chosen bulb will last between 850 and 1050 hours?

BIOLOGY **CONNECTION** In Exercises 44–46, use the following information.
According to a survey by the National Center for Health Statistics, the heights of adult women in the United States are normally distributed with a mean of 64 inches and a standard deviation of 2.7 inches.

44. What is the probability that three randomly selected women are all 58.6 inches or shorter?

45. What is the probability that five randomly selected women are all between the heights of 61.3 and 66.7 inches?

46. What is the probability that four randomly selected women are all 72.1 inches or taller?

🌎 **SAT SCORES** In Exercises 47–49, use the following information.

In 1998 scores on the mathematics section of the SAT (Scholastic Aptitude Test) were normally distributed with a mean of 512 and a standard deviation of 112. Scores on the English section of the SAT were normally distributed with a mean of 505 and a standard deviation of 111. 🔖 **DATA UPDATE** of SAT data at www.mcdougallittell.com

47. What is the probability that a randomly chosen student who took the SAT in 1998 scored at least 736 on the mathematics section and at least 727 on the English section? Assume the scores are independent.

48. What is the probability that five randomly chosen students who took the SAT in 1998 all scored at most 394 on the English section?

49. What is the probability that two randomly chosen students who took the SAT in 1998 both scored between 400 and 624 on the mathematics section?

🌎 **LEFT-HANDEDNESS** In Exercises 50–52, use the fact that approximately 9% of people are left-handed, and consider a high school with 1221 students.

50. What is the probability that at least 140 students are left-handed?

51. What is the probability that at most 100 students are left-handed?

52. What is the probability that between 80 and 130 students are left-handed?

🌎 **MYOPIA** In Exercises 53–55, use the fact that myopia, or nearsightedness, is a condition that affects approximately 25% of the adult population in the United States, and consider a random sample of 192 people.

53. What is the probability that 42 or more people are nearsighted?

54. What is the probability that between 36 and 60 people are nearsighted?

55. What is the probability that 66 or fewer people are nearsighted?

Test Preparation

56. MULTI-STEP PROBLEM In 1998 Ben took both the SAT (Scholastic Aptitude Test) and the ACT (American College Test). On the mathematics section of the SAT, he earned a score of 624. On the mathematics section of the ACT, he earned a score of 31. For the SAT the mean was 512 and the standard deviation was 112. For the ACT the mean was 21 and the standard deviation was 5.

   a. What percent of students did Ben outscore on the math section of the SAT?

   b. What percent of students did Ben outscore on the math section of the ACT?

   c. On which exam did Ben score better?

   d. *Writing* Explain how you could translate ACT scores such as 15, 20, 25, and 30 into equivalent SAT scores if you know the mean and standard deviation of each exam.

★ **Challenge**

57. 🖩 **NORMAL CURVE** A normal curve is defined by an equation whose general form is as follows:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}$$

Use a graphing calculator to draw a histogram of a binomial distribution consisting of $n = 20$ trials with probability $p = 0.5$ of success. Also graph the normal curve that approximates this binomial distribution. Then graph other binomial distributions and normal curves in which you change p but leave n constant. When is the normal curve a good approximation of a binomial distribution and when is it a poor approximation? Why?

MIXED REVIEW

EVALUATING EXPRESSIONS Evaluate the expression without using a calculator. **(Review 7.1 for 13.1)**

58. $9^{3/2}$

59. $256^{3/4}$

60. $49^{-1/2}$

61. $\sqrt[3]{125}$

62. $\sqrt{81}$

63. $\left(\sqrt[4]{625}\right)^2$

IDENTIFYING PARTS Write the equation of the ellipse in standard form (if not already). Then identify the vertices, co-vertices, and foci of the ellipse. **(Review 10.4)**

64. $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$

65. $\dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$

66. $\dfrac{x^2}{5} + \dfrac{y^2}{10} = 1$

67. $\dfrac{x^2}{6} + \dfrac{y^2}{21} = 1$

68. $4x^2 + 9y^2 = 36$

69. $10x^2 + 7y^2 = 70$

USING COMPLEMENTS Two six-sided dice are rolled. Find the probability of the given event. **(Review 12.4)**

70. The sum is not 4.

71. The sum is less than or equal to 10.

72. The sum is not 3 or 12.

73. The sum is greater than 3.

QUIZ 3

Calculate the probability of rolling a six-sided die 50 times and getting the given number of ones. **(Lesson 12.6)**

1. 0

2. 1

3. 8

4. 17

5. 25

6. 33

7. 42

8. 50

A binomial experiment consists of *n* trials with probability *p* of success on each trial. Draw a histogram of the binomial distribution that shows the probability of exactly *k* successes. Then find the most likely number of successes. **(Lesson 12.6)**

9. $n = 4, p = 0.3$

10. $n = 7, p = 0.5$

11. $n = 8, p = 0.6$

12. $n = 10, p = 0.33$

13. $n = 12, p = 0.48$

14. $n = 15, p = 0.21$

A normal distribution has a mean of 62 and a standard deviation of 4. Find the probability that a randomly selected *x*-value is in the given interval. **(Lesson 12.7)**

15. between 58 and 66

16. between 62 and 74

17. between 50 and 70

18. 62 or greater

19. 58 or less

20. 50 or less

21. 🌎 **WELL-BEING** A survey that asked people in the United States about their feelings of personal well-being found that 85% are generally happy. To test this finding you survey 26 people at random and find that 19 consider themselves generally happy. Would you reject the survey's findings? Explain. **(Lesson 12.6)**

22. 🌎 **DRINKING WATER** Approximately 64% of people in the United States think that the nation's water supply is safe to drink. A town has 625 people. What is the probability that 400 or more people think that the nation's water supply is safe to drink? **(Lesson 12.7)**