12.2 Combinations and the Binomial Theorem

What you should learn

GOAL 1 Use combinations to count the number of ways an event can happen, as applied in **Ex. 55**.

GOAL 2 Use the binomial theorem to expand a binomial that is raised to a power.

Why you should learn it

▼ To solve **real-life** problems, such as finding the number of different combinations of plays you can attend in **Example 3**.

REAL LIFE

GOAL 1 USING COMBINATIONS

In Lesson 12.1 you learned that order is important for some counting problems. For other counting problems, order is not important. For instance, in most card games the order in which your cards are dealt is not important. After your cards are dealt, reordering them does not change your card hand. These unordered groupings are called *combinations*. A **combination** is a selection of r objects from a group of n objects where the order is not important.

COMBINATIONS OF *n* OBJECTS TAKEN *r* AT A TIME

The number of combinations of r objects taken from a group of n distinct objects is denoted by ${}_nC_r$ and is given by:

$$ {}_nC_r = \frac{n!}{(n-r)! \cdot r!} $$

For instance, the number of combinations of 2 objects taken from a group of 5 objects is ${}_5C_2 = \frac{5!}{3! \cdot 2!} = 10$.

EXAMPLE 1 *Finding Combinations*

A standard deck of 52 playing cards has 4 suits with 13 different cards in each suit as shown.

a. If the order in which the cards are dealt is not important, how many different 5-card hands are possible?

b. In how many of these hands are all five cards of the same suit?

Standard 52-Card Deck

K ♠	K ♣	K ♦	K ♥
Q ♠	Q ♣	Q ♦	Q ♥
J ♠	J ♣	J ♦	J ♥
10 ♠	10 ♣	10 ♦	10 ♥
9 ♠	9 ♣	9 ♦	9 ♥
8 ♠	8 ♣	8 ♦	8 ♥
7 ♠	7 ♣	7 ♦	7 ♥
6 ♠	6 ♣	6 ♦	6 ♥
5 ♠	5 ♣	5 ♦	5 ♥
4 ♠	4 ♣	4 ♦	4 ♥
3 ♠	3 ♣	3 ♦	3 ♥
2 ♠	2 ♣	2 ♦	2 ♥
A ♠	A ♣	A ♦	A ♥

SOLUTION

a. The number of ways to choose 5 cards from a deck of 52 cards is:

$$ {}_{52}C_5 = \frac{52!}{47! \cdot 5!} $$

$$ = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!} \cdot 5!} $$

$$ = 2{,}598{,}960 $$

b. For all five cards to be the same suit, you need to choose 1 of the 4 suits and then 5 of the 13 cards in the suit. So, the number of possible hands is:

$$ {}_4C_1 \cdot {}_{13}C_5 = \frac{4!}{3! \cdot 1!} \cdot \frac{13!}{8! \cdot 5!} = \frac{4 \cdot \cancel{3!}}{\cancel{3!} \cdot 1!} \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!} \cdot 5!} = 5148 $$

When finding the number of ways both an event A *and* an event B can occur, you need to multiply (as you did in part (b) of Example 1). When finding the number of ways that an event A *or* an event B can occur, you add instead.

Menu Choices

EXAMPLE 2 *Deciding to Multiply or Add*

A restaurant serves omelets that can be ordered with any of the ingredients shown.

a. Suppose you want *exactly* 2 vegetarian ingredients and 1 meat ingredient in your omelet. How many different types of omelets can you order?

b. Suppose you can afford *at most* 3 ingredients in your omelet. How many different types of omelets can you order?

Omelets $3.00
(plus $.50 for each ingredient)

Vegetarian	Meat
green pepper	ham
red pepper	bacon
onion	sausage
mushroom	steak
tomato	
cheese	

SOLUTION

a. You can choose 2 of 6 vegetarian ingredients and 1 of 4 meat ingredients. So, the number of possible omelets is:

$$_6C_2 \cdot {}_4C_1 = \frac{6!}{4! \cdot 2!} \cdot \frac{4!}{3! \cdot 1!} = 15 \cdot 4 = 60$$

b. You can order an omelet with 0, 1, 2, or 3 ingredients. Because there are 10 items to choose from, the number of possible omelets is:

$$_{10}C_0 + {}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 = 1 + 10 + 45 + 120 = 176$$

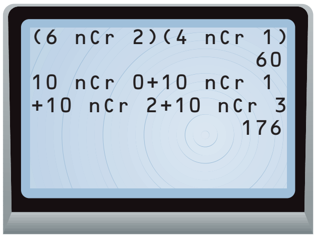
. . . .

STUDENT HELP

INTERNET

KEYSTROKE HELP
Visit our Web site www.mcdougallittell.com to see keystrokes for several models of calculators.

Some calculators have special keys to evaluate combinations. The solution to Example 2 is shown.



Counting problems that involve phrases like "at least" or "at most" are sometimes easier to solve by subtracting possibilities you do not want from the total number of possibilities.

Theater

EXAMPLE 3 *Subtracting Instead of Adding*

A theater is staging a series of 12 different plays. You want to attend *at least* 3 of the plays. How many different combinations of plays can you attend?

SOLUTION

You want to attend 3 plays, or 4 plays, or 5 plays, and so on. So, the number of combinations of plays you can attend is $_{12}C_3 + {}_{12}C_4 + {}_{12}C_5 + \cdots + {}_{12}C_{12}$.

Instead of adding these combinations, it is easier to use the following reasoning. For each of the 12 plays, you can choose to attend or not attend the play, so there are 2^{12} total combinations. If you attend at least 3 plays you do not attend only 0, 1, or 2 plays. So, the number of ways you can attend at least 3 plays is:

$$2^{12} - ({}_{12}C_0 + {}_{12}C_1 + {}_{12}C_2) = 4096 - (1 + 12 + 66) = 4017$$

FOCUS ON PEOPLE

BLAISE PASCAL developed his arithmetic triangle in 1653. The following year he and fellow mathematician Pierre Fermat outlined the foundations of probability theory.

GOAL 2 USING THE BINOMIAL THEOREM

If you arrange the values of ${}_nC_r$ in a triangular pattern in which each row corresponds to a value of n, you get what is called **Pascal's triangle**. It is named after the famous French mathematician Blaise Pascal (1623–1662).

$$\begin{array}{ccccccccccc}
 & & & & & {}_0C_0 & & & & & \\
 & & & & {}_1C_0 & & {}_1C_1 & & & & \\
 & & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & & \\
 & & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & & \\
 & {}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4 & \\
{}_5C_0 & & {}_5C_1 & & {}_5C_2 & & {}_5C_3 & & {}_5C_4 & & {}_5C_5
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Pascal's triangle has many interesting patterns and properties. For instance, each number other than 1 is the sum of the two numbers directly above it.

ACTIVITY Developing Concepts — Investigating Pascal's Triangle

1. Expand each expression. Write the terms of each expanded expression so that the powers of a decrease.

   a. $(a + b)^2$ **b.** $(a + b)^3$ **c.** $(a + b)^4$

2. Describe the relationship between the coefficients in parts (a), (b), and (c) of **Step 1** and the rows of Pascal's triangle.

3. Describe any patterns in the exponents of a and the exponents of b.

In the activity you may have discovered the following result, which is called the **binomial theorem**. This theorem describes the coefficients in the expansion of the binomial $a + b$ raised to the nth power.

THE BINOMIAL THEOREM

The binomial expansion of $(a + b)^n$ for any positive integer n is:

$$(a + b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_na^0b^n$$
$$= \sum_{r=0}^{n} {}_nC_ra^{n-r}b^r$$

EXAMPLE 4 *Expanding a Power of a Simple Binomial Sum*

Expand $(x + 2)^4$.

SOLUTION

$$(x + 2)^4 = {}_4C_0x^42^0 + {}_4C_1x^32^1 + {}_4C_2x^22^2 + {}_4C_3x^12^3 + {}_4C_4x^02^4$$
$$= (1)(x^4)(1) + (4)(x^3)(2) + (6)(x^2)(4) + (4)(x)(8) + (1)(1)(16)$$
$$= x^4 + 8x^3 + 24x^2 + 32x + 16$$

STUDENT HELP

Study Tip
You can calculate combinations using either Pascal's triangle or the formula on p. 708.

EXAMPLE 5 Expanding a Power of a Binomial Sum

Expand $(u + v^2)^3$.

SOLUTION

$$(u + v^2)^3 = {}_3C_0u^3(v^2)^0 + {}_3C_1u^2(v^2)^1 + {}_3C_2u^1(v^2)^2 + {}_3C_3u^0(v^2)^3$$
$$= u^3 + 3u^2v^2 + 3uv^4 + v^6$$

. . . .

To expand a power of a binomial difference, you can rewrite the binomial as a sum. The resulting expansion will have terms whose signs alternate between + and −.

EXAMPLE 6 Expanding a Power of a Simple Binomial Difference

Expand $(x - y)^5$.

SOLUTION

$$(x - y)^5 = [x + (-y)]^5$$
$$= {}_5C_0x^5(-y)^0 + {}_5C_1x^4(-y)^1 + {}_5C_2x^3(-y)^2 + {}_5C_3x^2(-y)^3 + {}_5C_4x^1(-y)^4 + {}_5C_5x^0(-y)^5$$
$$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

EXAMPLE 7 Expanding a Power of a Binomial Difference

Expand $(5 - 2a)^4$.

SOLUTION

$$(5 - 2a)^4 = [5 + (-2a)]^4$$
$$= {}_4C_05^4(-2a)^0 + {}_4C_15^3(-2a)^1 + {}_4C_25^2(-2a)^2 + {}_4C_35^1(-2a)^3 + {}_4C_45^0(-2a)^4$$
$$= (1)(625)(1) + (4)(125)(-2a) + (6)(25)(4a^2) + (4)(5)(-8a^3) + (1)(1)(16a^4)$$
$$= 625 - 1000a + 600a^2 - 160a^3 + 16a^4$$

EXAMPLE 8 Finding a Coefficient in an Expansion

Find the coefficient of x^4 in the expansion of $(2x - 3)^{12}$.

SOLUTION From the binomial theorem you know the following:

$$(2x - 3)^{12} = \sum_{r=0}^{12} {}_{12}C_r(2x)^{12-r}(-3)^r$$

The term that has x^4 is ${}_{12}C_8(2x)^4(-3)^8 = (495)(16x^4)(6561) = 51{,}963{,}120x^4$.

▶ The coefficient is 51,963,120.

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain the difference between a permutation and a combination.

Concept Check ✓

2. Describe a situation in which to find the total number of possibilities you would (**a**) add two combinations and (**b**) multiply two combinations.
3. Write the expansions for $(x + y)^4$ and $(x - y)^4$. How are they similar? How are they different?
4. **ERROR ANALYSIS** What error was made in the calculation of $_{10}C_6$? Explain.

$$_{10}C_6 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7$$

Skill Check ✓

Find the number of combinations of *n* objects taken *r* at a time.

5. $n = 8, r = 2$ 6. $n = 6, r = 5$ 7. $n = 5, r = 1$ 8. $n = 9, r = 9$

Expand the power of the binomial.

9. $(x + y)^3$ 10. $(x + 1)^4$ 11. $(2x + 4)^3$ 12. $(2x + 3y)^5$

13. $(x - y)^5$ 14. $(x - 2)^3$ 15. $(3x - 1)^4$ 16. $(4x - 4y)^3$

17. Complete this equation:

$$(x + 3y)^5 = x^5 + 15x^4y + 90x^3y^2 + \underline{?}\ x^2y^3 + 405xy^4 + \underline{?}\ y^5$$

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 956.

COMBINATIONS **Find the number of combinations.**

18. $_{10}C_2$ 19. $_8C_5$ 20. $_5C_2$ 21. $_8C_6$

22. $_{12}C_4$ 23. $_{12}C_{12}$ 24. $_{14}C_6$ 25. $_{11}C_3$

CARD HANDS **In Exercises 26–30, find the number of possible 5-card hands that contain the cards specified.**

26. 5 face cards (either kings, queens, or jacks)
27. 4 aces and 1 other card
28. 1 ace and 4 other cards (none of which are aces)
29. 2 aces and 3 kings
30. 4 of one kind (kings, queens, and so on) and 1 of a different kind
31. **PASCAL'S TRIANGLE** Copy Pascal's triangle on page 710 and add the rows for $n = 6$ and $n = 7$ to it.

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 18–30, 47, 48
Example 2: Exs. 49–52
Example 3: Exs. 53, 54
Examples 4–7: Exs. 31–43
Example 8: Exs. 44–46

PASCAL'S TRIANGLE **Use the rows of Pascal's triangle from Exercise 31 to write the binomial expansion.**

32. $(x + 4)^6$ 33. $(x - 3y)^6$ 34. $(x^2 + y)^7$ 35. $(2x - y^3)^7$

BINOMIAL THEOREM **Use the binomial theorem to write the binomial expansion.**

36. $(x - 2)^3$ 37. $(x + 4)^5$ 38. $(x + 3y)^4$ 39. $(2x - y)^6$

40. $(x^3 + 3)^5$ 41. $(3x^2 - 3)^4$ 42. $(2x - y^2)^7$ 43. $(x^3 + y^2)^3$

FOCUS ON APPLICATIONS

REAL LIFE **CARS** A 1998 survey showed that of 7 basic car colors, white is the most popular color for full-size cars with 18.8% of the vote. Green came in second with 16.4% of the vote.

INTERNET **APPLICATION LINK**
www.mcdougallittell.com

44. Find the coefficient of x^5 in the expansion of $(x - 3)^7$.

45. Find the coefficient of x^4 in the expansion of $(x + 2)^8$.

46. Find the coefficient of x^6 in the expansion of $(x^2 + 4)^{10}$.

47. **NOVELS** Your English teacher has asked you to select 3 novels from a list of 10 to read as an independent project. In how many ways can you choose which books to read?

48. **GAMES** Your friend is having a party and has 15 games to choose from. There is enough time to play 4 games. In how many ways can you choose which games to play?

49. **CARS** You are buying a new car. There are 7 different colors to choose from and 10 different types of optional equipment you can buy. You can choose only 1 color for your car and can afford only 2 of the options. How many combinations are there for your car?

50. **ART CONTEST** There are 6 artists each presenting 5 works of art in an art contest. The 4 works judged best will be displayed in a local gallery. In how many ways can these 4 works all be chosen from the same artist's collection?

51. **LOGICAL REASONING** Look back at Example 2. Suppose you can afford at most 7 ingredients. How many different types of omelets can you order?

52. **AMUSEMENT PARKS** An amusement park has 20 different rides. You want to ride at least 15 of them. How many different combinations of rides can you go on?

53. **FISH** From the list of different species of fish shown, an aquarium enthusiast is interested in knowing how compatible any group of 3 or more different species are. How many different combinations are there to consider?

54. **CONCERTS** A summer concert series has 12 different performing artists. You decide to attend at least 4 of the concerts. How many different combinations of concerts can you attend?

CRITICAL THINKING **Decide whether the problem requires combinations or permutations to find the answer. Then solve the problem.**

55. **MARCHING BAND** Eight members of a school marching band are auditioning for 3 drum major positions. In how many ways can students be chosen to be drum majors?

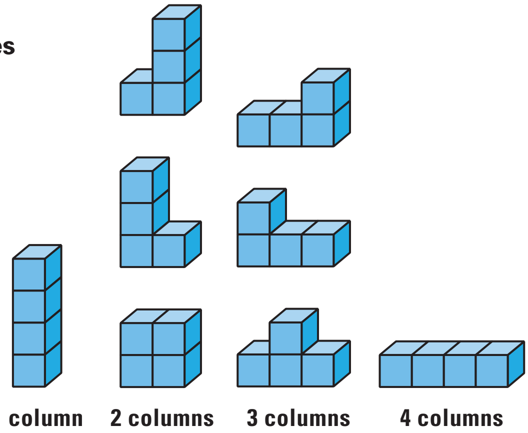
56. **YEARBOOK** Your school yearbook has an editor-in-chief and an assistant editor-in-chief. The staff of the yearbook has 15 students. In how many ways can students be chosen for these 2 positions?

57. **RELAY RACES** A relay race has 4 runners who run different parts of the race. There are 16 students on your track team. In how many ways can your coach select students to compete in the race?

58. **COLLEGE COURSES** You must take 6 elective classes to meet your graduation requirements for college. There are 12 classes that you are interested in. In how many ways can you select your elective classes?

59. **CRITICAL THINKING** Write an equation that relates ${}_nP_r$ and ${}_nC_r$.

STACKING CUBES **In Exercises 60–63, use the diagram shown which illustrates the different ways to stack four cubes.**



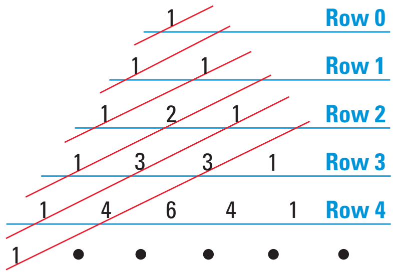
60. Sketch the different ways to stack three cubes.

61. Sketch the different ways to stack five cubes.

62. How does the number of ways to stack three, four, and five cubes relate to Pascal's triangle?

63. In how many different ways can you stack ten cubes?

PASCAL'S TRIANGLE **In Exercises 64–66, use the diagram of Pascal's triangle shown.**



64. What is the sum of the numbers in row n of Pascal's triangle? Explain.

65. What is the sum of the numbers in rows 0 through 20 of Pascal's triangle?

66. **LOGICAL REASONING** Describe the pattern formed by the sums of the numbers along the diagonal segments of Pascal's triangle.

Test Preparation

67. **MULTI-STEP PROBLEM** A group of 20 high school students is volunteering to help elderly members of their community. Each student will be assigned a job based on requests received for help. There are 8 requests for raking leaves, 7 requests for running errands, and 5 requests for washing windows.

a. One way to count the number of possible job assignments is to find the number of permutations of 8 *L*'s (for "leaves"), 7 *E*'s (for "errands"), and 5 *W*'s (for "windows"). Use this method to write the number of possible job assignments first as an expression involving factorials and then as a simple number.

b. Another way to count the number of possible job assignments is to first choose the 8 students who will rake leaves, then choose the 7 students who will run errands from the students who remain, and then choose the 5 students who will wash windows from the students who still remain. Use this method to write the number of possible job assignments first as an expression involving factorials and then as a simple number.

c. *Writing* How do the answers to parts (a) and (b) compare to each other? Explain why this makes sense.

★ **Challenge**

COMBINATORIAL IDENTITIES **Verify the identity.**

68. ${}_nC_0 = 1$

69. ${}_nC_n = 1$

70. ${}_nC_1 = {}_nP_1$

71. ${}_nC_r = {}_nC_{n-r}$

72. ${}_nC_r \cdot {}_rC_m = {}_nC_m \cdot {}_{n-m}C_{r-m}$

73. ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

MIXED REVIEW

FINDING AREA **Find the area of the figure.** **(Skills Review, p. 914)**

74. Circle with radius 18 centimeters

75. Rectangle with sides 9.5 inches and 11.3 inches

76. Triangle with base 13 feet and height 9 feet

77. Trapezoid with bases 10 meters and 13 meters, and height 27 meters

GRAPHING **Graph the equation of the hyperbola.** **(Review 10.5)**

78. $\frac{x^2}{25} - \frac{y^2}{144} = 1$ **79.** $\frac{y^2}{100} - \frac{x^2}{36} = 1$ **80.** $x^2 - \frac{49y^2}{16} = 1$

81. $\frac{y^2}{4} - \frac{x^2}{9} = 9$ **82.** $64y^2 - x^2 = 64$ **83.** $9x^2 - 4y^2 = 144$

WRITING RULES **Decide whether the sequence is arithmetic or geometric. Then write a rule for the *n*th term.** **(Review 11.2, 11.3)**

84. 3, 9, 27, 81, 243, . . . **85.** 3, 10, 17, 24, 31, . . . **86.** 2, 10, 50, 250, 1250, . . .

87. 1, −2, 4, −8, 16, . . . **88.** 8, 6, 4, 2, 0, . . . **89.** −10, −5, 0, 5, 10, . . .

90. **POTTERY** A potter has 70 pounds of clay and 40 hours to make soup bowls and dinner plates to sell at a craft fair. A soup bowl uses 3 pounds of clay and a dinner plate uses 4 pounds of clay. It takes 3 hours to make a soup bowl and 1 hour to make a dinner plate. If the profit on a soup bowl is \$25 and the profit on a dinner plate is \$20, how many bowls and plates should the potter make in order to maximize profit? **(Review 3.4)**

QUIZ 1

Self-Test for Lessons 12.1 and 12.2

Find the number of distinguishable permutations of the letters in the word. **(Lesson 12.1)**

1. POP **2.** JUNE **3.** IDAHO **4.** KANSAS

5. WYOMING **6.** THURSDAY **7.** SEPTEMBER **8.** CALIFORNIA

Write the binomial expansion. **(Lesson 12.2)**

9. $(x + y)^6$ **10.** $(x + 2)^4$ **11.** $(x - 2y)^5$ **12.** $(3x - 4y)^3$

13. $(x^2 + 3y)^4$ **14.** $(4x^2 - 2)^6$ **15.** $(x^3 - y^3)^3$ **16.** $(2x^4 + 5y^2)^5$

17. Find the coefficient of x^3 in the expansion of $(x + 3)^5$. **(Lesson 12.2)**

18. Find the coefficient of y^4 in the expansion of $(5 - y^2)^3$. **(Lesson 12.2)**

19. **RESTAURANTS** You are eating dinner at a restaurant. The restaurant offers 6 appetizers, 12 main dishes, 6 side orders, and 8 desserts. If you order one of each of these, how many different dinners can you order? **(Lesson 12.1)**

20. **FLOWERS** You are buying a flower arrangement. The florist has 12 types of flowers and 6 types of vases. If you can afford exactly 3 types of flowers and need only 1 vase, how many different arrangements can you buy? **(Lesson 12.2)**