12.1 The Fundamental Counting Principle and Permutations

What you should learn

GOAL 1 Use the fundamental counting principle to count the number of ways an event can happen.

GOAL 2 Use permutations to count the number of ways an event can happen, as applied in **Ex. 62**.

Why you should learn it

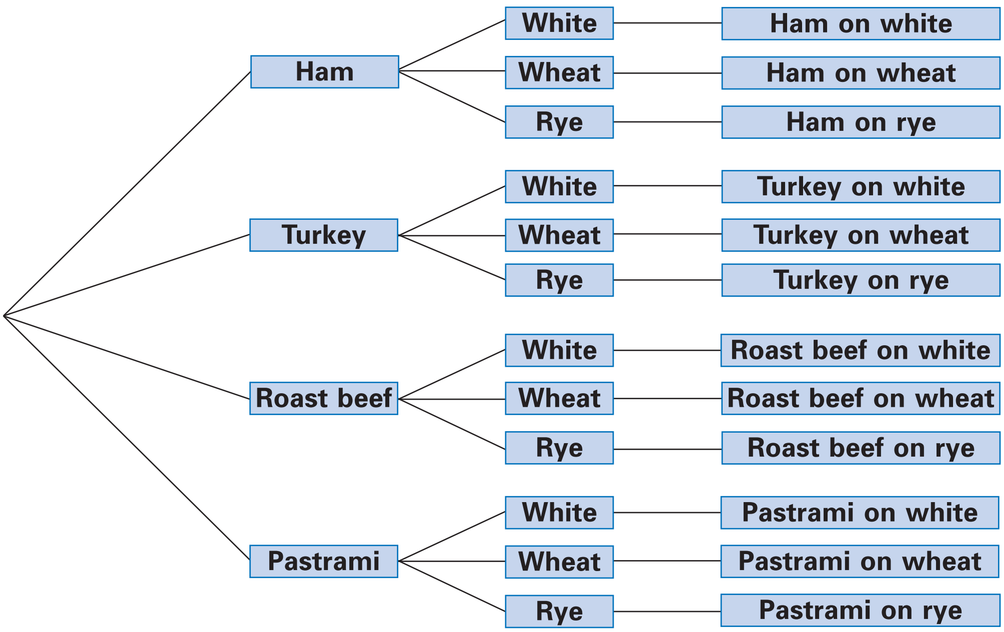
▼ To find the number of ways a **real-life** event can happen, such as the number of ways skiers can finish in an aerial competition in **Example 3**.

REAL LIFE

GOAL 1 THE FUNDAMENTAL COUNTING PRINCIPLE

In many real-life problems you want to count the number of possibilities. For instance, suppose you own a small deli. You offer 4 types of meat (ham, turkey, roast beef, and pastrami) and 3 types of bread (white, wheat, and rye). How many choices do your customers have for a meat sandwich?

One way to answer this question is to use a *tree diagram*, as shown below. From the list on the right you can see that there are 12 choices.



Another way to count the number of possible sandwiches is to use the *fundamental counting principle*. Because you have 4 choices for meat and 3 choices for bread, the total number of choices is $4 \cdot 3 = 12$.

FUNDAMENTAL COUNTING PRINCIPLE

TWO EVENTS If one event can occur in m ways and another event can occur in n ways, then the number of ways that *both* events can occur is $m \cdot n$.

For instance, if one event can occur in 2 ways and another event can occur in 5 ways, then both events can occur in $2 \cdot 5 = 10$ ways.

THREE OR MORE EVENTS The fundamental counting principle can be extended to three or more events. For example, if three events can occur in m, n, and p ways, then the number of ways that *all* three events can occur is $m \cdot n \cdot p$.

For instance, if three events can occur in 2, 5, and 7 ways, then all three events can occur in $2 \cdot 5 \cdot 7 = 70$ ways.

FOCUS ON CAREERS

POLICE DETECTIVE

A police detective is an officer who collects facts and evidence for criminal cases. Part of a detective's duties may include helping witnesses identify suspects.

CAREER LINK
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EXAMPLE 1 *Using the Fundamental Counting Principle*

CRIMINOLOGY Police use photographs of various facial features to help witnesses identify suspects. One basic identification kit contains 195 hairlines, 99 eyes and eyebrows, 89 noses, 105 mouths, and 74 chins and cheeks.

▶ Source: *Readers' Digest: How In The World?*

a. The developer of the identification kit claims that it can produce billions of different faces. Is this claim correct?

b. A witness can clearly remember the hairline and the eyes and eyebrows of a suspect. How many different faces can be produced with this information?

SOLUTION

a. You can use the fundamental counting principle to find the total number of different faces.

$$\text{Number of faces} = 195 \cdot 99 \cdot 89 \cdot 105 \cdot 74 = 13{,}349{,}986{,}650$$

▶ The developer's claim is correct since the kit can produce over 13 billion faces.

b. Because the witness clearly remembers the hairline and the eyes and eyebrows, there is only 1 choice for each of these features. You can use the fundamental counting principle to find the number of different faces.

$$\text{Number of faces} = 1 \cdot 1 \cdot 89 \cdot 105 \cdot 74 = 691{,}530$$

▶ The number of faces that can be produced has been reduced to 691,530.

License Plates

EXAMPLE 2 *Using the Fundamental Counting Principle with Repetition*

The standard configuration for a New York license plate is 3 digits followed by 3 letters.

▶ Source: New York State Department of Motor Vehicles

a. How many different license plates are possible if digits and letters can be repeated?

b. How many different license plates are possible if digits and letters cannot be repeated?

SOLUTION

a. There are 10 choices for each digit and 26 choices for each letter. You can use the fundamental counting principle to find the number of different plates.

$$\text{Number of plates} = 10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 17{,}576{,}000$$

▶ The number of different license plates is 17,576,000.

b. If you cannot repeat digits there are still 10 choices for the first digit, but then only 9 remaining choices for the second digit and only 8 remaining choices for the third digit. Similarly, there are 26 choices for the first letter, 25 choices for the second letter, and 24 choices for the third letter. You can use the fundamental counting principle to find the number of different plates.

$$\text{Number of plates} = 10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 = 11{,}232{,}000$$

▶ The number of different license plates is 11,232,000.

GOAL 2 USING PERMUTATIONS

> **STUDENT HELP**
>
> **Study Tip**
> Recall from Lesson 11.5 that $n!$ is read as "n factorial." Also note that $0! = 1$ and $1! = 1$.

An ordering of n objects is a **permutation** of the objects. For instance, there are six permutations of the letters A, B, and C: ABC, ACB, BAC, BCA, CAB, CBA.

The fundamental counting principle can be used to determine the number of permutations of n objects. For instance, you can find the number of ways you can arrange the letters A, B, and C by multiplying. There are 3 choices for the first letter, 2 choices for the second letter, and 1 choice for the third letter, so there are $3 \cdot 2 \cdot 1 = 6$ ways to arrange the letters.

In general, the number of permutations of n distinct objects is:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

REAL LIFE Sports

EXAMPLE 3 *Finding the Number of Permutations*

Twelve skiers are competing in the final round of the Olympic freestyle skiing aerial competition.

a. In how many different ways can the skiers finish the competition? (Assume there are no ties.)

b. In how many different ways can 3 of the skiers finish first, second, and third to win the gold, silver, and bronze medals?

SOLUTION

a. There are 12! different ways that the skiers can finish the competition.

$$12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 479{,}001{,}600$$

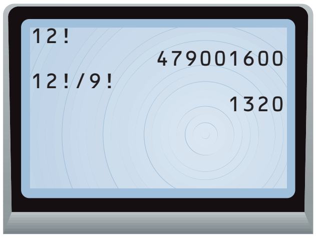
b. Any of the 12 skiers can finish first, then any of the remaining 11 skiers can finish second, and finally any of the remaining 10 skiers can finish third. So, the number of ways that the skiers can win the medals is:

$$12 \cdot 11 \cdot 10 = 1320$$

> **STUDENT HELP**
>
> INTERNET **KEYSTROKE HELP**
> Visit our Web site www.mcdougallittell.com to see keystrokes for several models of calculators.

Some calculators have special keys to evaluate factorials. The solution to Example 3 is shown.

The number in part (b) of Example 3 is called the number of permutations of 12 objects taken 3 at a time, is denoted by ${}_{12}P_3$, and is given by $\frac{12!}{(12-3)!}$.



> **STUDENT HELP**
>
> **Derivations**
> For a derivation of the formula for the permutation of n objects taken r at a time, see p. 899.

PERMUTATIONS OF n OBJECTS TAKEN r AT A TIME

The number of permutations of r objects taken from a group of n distinct objects is denoted by ${}_nP_r$ and is given by:

$${}_nP_r = \frac{n!}{(n-r)!}$$

REAL LIFE
College Visits

EXAMPLE 4 *Finding Permutations of n Objects Taken r at a Time*

You are considering 10 different colleges. Before you decide to apply to the colleges, you want to visit some or all of them. In how many orders can you visit (**a**) 6 of the colleges and (**b**) all 10 colleges?

SOLUTION

a. The number of permutations of 10 objects taken 6 at a time is:

$${}_{10}P_6 = \frac{10!}{(10-6)!} = \frac{10!}{4!} = \frac{3{,}628{,}800}{24} = 151{,}200$$

b. The number of permutations of 10 objects taken 10 at a time is:

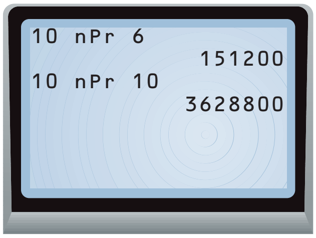
$${}_{10}P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = 10! = 3{,}628{,}800$$

....

STUDENT HELP

INTERNET
KEYSTROKE HELP
Visit our Web site www.mcdougallittell.com to see keystrokes for several models of calculators.

Some calculators have special keys that are programmed to evaluate ${}_nP_r$. The solution to Example 4 is shown.



So far you have been finding permutations of *distinct* objects. If some of the objects are repeated, then some of the permutations are not distinguishable. For instance, of the six ways to order the letters M, O, and M—

M O M   O M M   M M O

M O M   O M M   M M O

—only three are distinguishable without color: MOM, OMM, and MMO. In this case, the number of permutations is $\frac{3!}{2!} = \frac{6}{2} = 3$, not $3! = 6$.

PERMUTATIONS WITH REPETITION

The number of distinguishable permutations of n objects where one object is repeated q_1 times, another is repeated q_2 times, and so on is:

$$\frac{n!}{q_1! \cdot q_2! \cdot \ldots \cdot q_k!}$$

EXAMPLE 5 *Finding Permutations with Repetition*

Find the number of distinguishable permutations of the letters in (**a**) OHIO and (**b**) MISSISSIPPI.

SOLUTION

a. OHIO has 4 letters of which O is repeated 2 times. So, the number of distinguishable permutations is $\frac{4!}{2!} = \frac{24}{2} = 12$.

b. MISSISSIPPI has 11 letters of which I is repeated 4 times, S is repeated 4 times, and P is repeated 2 times. So, the number of distinguishable permutations is

$$\frac{11!}{4! \cdot 4! \cdot 2!} = \frac{39{,}916{,}800}{24 \cdot 24 \cdot 2} = 34{,}650.$$

GUIDED PRACTICE

Vocabulary Check ✓

1. What is a permutation of n objects?

Concept Check ✓

2. Explain how the fundamental counting principle can be used to justify the formula for the number of permutations of n distinct objects.
3. Rita found the number of distinguishable permutations of the letters in OHIO by evaluating the expression $\frac{4!}{2! \cdot 1! \cdot 1!}$. Does this method give the same answer as in part (a) of Example 5? Explain.
4. **ERROR ANALYSIS** Explain the error in calculating how many three-digit numbers from 000 to 999 have only even digits.

> ~~Number of 3-digit numbers with only even digits = 5 • 4 • 3 = 60~~

Skill Check ✓

Find the number of permutations of *n* distinct objects.

5. $n = 2$
6. $n = 6$
7. $n = 1$
8. $n = 4$

Find the number of permutations of *n* objects taken *r* at a time.

9. $n = 6, r = 3$
10. $n = 5, r = 1$
11. $n = 3, r = 3$
12. $n = 10, r = 2$

Find the number of permutations of *n* objects where one or more objects are repeated the given number of times.

13. 7 objects with one object repeated 4 times
14. 5 objects with one object repeated 3 times and a second object repeated 2 times

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 956.

FUNDAMENTAL COUNTING PRINCIPLE **Each event can occur in the given number of ways. Find the number of ways all of the events can occur.**

15. Event 1: 1 way, Event 2: 3 ways
16. Event 1: 3 ways, Event 2: 5 ways
17. Event 1: 2 ways, Event 2: 4 ways, Event 3: 5 ways
18. Event 1: 4 ways, Event 2: 6 ways, Event 3: 9 ways, Event 4: 7 ways

LICENSE PLATES **For the given configuration, determine how many different license plates are possible if (a) digits and letters can be repeated, and (b) digits and letters cannot be repeated.**

19. 3 letters followed by 3 digits
20. 2 digits followed by 4 letters
21. 4 digits followed by 2 letters
22. 5 letters followed by 1 digit

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 15–18, 55, 56
Example 2: Exs. 19–22, 57
Example 3: Exs. 23–30, 39–46, 59, 60
Example 4: Exs. 31–38, 61
Example 5: Exs. 47–54, 62, 63

FACTORIALS **Evaluate the factorial.**

23. 8!
24. 5!
25. 10!
26. 9!
27. 0!
28. 7!
29. 3!
30. 12!

PERMUTATIONS **Find the number of permutations.**

31. ${}_3P_3$
32. ${}_5P_2$
33. ${}_2P_1$
34. ${}_7P_6$
35. ${}_8P_5$
36. ${}_9P_4$
37. ${}_{12}P_3$
38. ${}_{16}P_0$

FOCUS ON APPLICATIONS

COMPUTER SECURITY On the Internet there are three main ways to secure a site: restrict which addresses can access the site, use public key cryptography, or require a user name and password.

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 61.

PERMUTATIONS WITHOUT REPETITION **Find the number of distinguishable permutations of the letters in the word.**

39. HI **40.** JET **41.** IOWA **42.** TEXAS

43. PENCIL **44.** FLORIDA **45.** MAGNETIC **46.** GOLDFINCH

PERMUTATIONS WITH REPETITION **Find the number of distinguishable permutations of the letters in the word.**

47. DAD **48.** PUPPY **49.** OREGON **50.** LETTER

51. ALGEBRA **52.** ALABAMA **53.** MISSOURI **54.** CONNECTICUT

55. **STEREO** You are going to set up a stereo system by purchasing separate components. In your price range you find 5 different receivers, 8 different compact disc players, and 12 different speaker systems. If you want one of each of these components, how many different stereo systems are possible?

56. **PIZZA** A pizza shop runs a special where you can buy a large pizza with one cheese, one vegetable, and one meat for $9.00. You have a choice of 7 cheeses, 11 vegetables, and 6 meats. Additionally, you have a choice of 3 crusts and 2 sauces. How many different variations of the pizza special are possible?

57. **COMPUTER SECURITY** To keep computer files secure, many programs require the user to enter a password. The shortest allowable passwords are typically six characters long and can contain both numbers and letters. How many six-character passwords are possible if (**a**) characters can be repeated and (**b**) characters cannot be repeated?

58. **CRITICAL THINKING** Simplify the formula for ${}_nP_r$ when $r = 0$. Explain why this result makes sense.

59. **CLASS SEATING** A particular classroom has 24 seats and 24 students. Assuming the seats are not moved, how many different seating arrangements are possible? Write your answer in scientific notation.

60. **RINGING BELLS** "Ringing the changes" is a process where the bells in a tower are rung in all possible permutations. Westminster Abbey has 10 bells in its tower. In how many ways can its bells be rung?

61. **PLAY AUDITIONS** Auditions are being held for the play shown. How many ways can the roles be assigned if (**a**) 6 people audition and (**b**) 9 people audition?

62. **WINDOW DISPLAY** A music store wants to display 3 identical keyboards, 2 identical trumpets, and 2 identical guitars in its store window. How many distinguishable displays are possible?

63. **DOG SHOW** In a dog show how many ways can 3 Chihuahuas, 5 Labradors, 4 poodles, and 3 beagles line up in front of the judges if the dogs of the same breed are considered identical?

64. **CRITICAL THINKING** Find the number of permutations of n objects taken $n - 1$ at a time for any positive integer n. Compare this answer with the number of permutations of all n objects. Does this make sense? Explain.

Test Preparation

QUANTITATIVE COMPARISON **In Exercises 65 and 66, choose the statement that is true about the given quantities.**

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

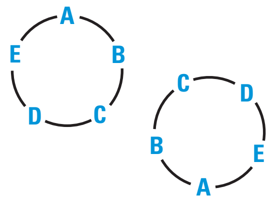
Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
65.	${}_5P_1$	5!
66.	The number of permutations of 12 objects taken 7 at a time	The number of permutations of 12 objects, one of which is repeated 5 times

★ Challenge

67. CIRCULAR PERMUTATIONS You have learned that $n!$ represents the number of ways that n objects can be placed in a *linear* order, where it matters which object is placed first. Now consider *circular* permutations, where objects are placed in a circle so it does *not* matter which object is placed first. Find a formula for the number of permutations of n objects placed in clockwise order around a circle when only the relative order of the objects matters. Explain how you derived your formula.



This is the same permutation.

EXTRA CHALLENGE
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MIXED REVIEW

SPECIAL PRODUCTS **Find the product.** **(Review 6.3 for 12.2)**

68. $(x + 9)(x - 9)$ **69.** $(x^2 + 2)^2$ **70.** $(2x - 1)^3$

71. $(4x + 5)(4x - 5)$ **72.** $(2y + 3x)^2$ **73.** $(8y - x)^2$

GRAPHING **Graph the equation of the parabola.** **(Review 10.2)**

74. $y^2 = 8x$ **75.** $x^2 = -10y$ **76.** $y^2 = -4x$

77. $x^2 = 26y$ **78.** $x + 14y^2 = 0$ **79.** $y - 2x^2 = 0$

FINDING SUMS **Find the sum of the infinite geometric series if there is one.** **(Review 11.4)**

80. $\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$ **81.** $\sum_{n=0}^{\infty} -4\left(\frac{1}{4}\right)^n$ **82.** $\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^n$

83. $\sum_{n=1}^{\infty} 2\left(\frac{7}{5}\right)^{n-1}$ **84.** $\sum_{n=0}^{\infty} -5\left(\frac{1}{8}\right)^n$ **85.** $\sum_{n=1}^{\infty} \frac{1}{2}(0.3)^{n-1}$

86. **SCIENCE CONNECTION** Ohm's law states that the resistance R (in ohms) of a conductor varies directly with the potential difference V (in volts) between two points and inversely with the current I (in amperes). The constant of variation is 1. What is the resistance of a light bulb if there is a current of 0.80 ampere when the potential difference across the bulb is 120 volts? **(Review 9.1)**