CHAPTER 11

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Use summation notation to write a series. **(11.1)**	Express the number of oranges in a stack. **(p. 654)**
Find terms of sequences.	
• defined by explicit rules **(11.1)**	Find angle measures at the tips of a star. **(p. 656)**
• defined by recursive rules **(11.5)**	Find the number of fish in a stocked lake. **(p. 683)**
Graph and classify sequences. **(11.1–11.3)**	Compare the revenues of two companies. **(p. 672)**
Write rules for nth terms of sequences.	
• given some terms **(11.1–11.3)**	Model the minimum number of moves in the Tower of Hanoi puzzle. **(p. 656)**
• arithmetic sequences **(11.2)**	Model the number of seats in a concert hall. **(p. 662)**
• geometric sequences **(11.3)**	Model the number of matches in a tennis tournament. **(p. 671)**
Find sums of series.	
• by adding terms or using formulas **(11.1)**	Find the number of tennis balls in a stack. **(p. 656)**
• finite arithmetic series **(11.2)**	Find the number of cells in a honeycomb. **(p. 664)**
• finite geometric series **(11.3)**	Find the cost of cellular telephone service. **(p. 669)**
• infinite geometric series **(11.4)**	Find the amount of money spent by tourists who receive a tourist brochure. **(p. 679)**
Write recursive rules for sequences. **(11.5)**	Model the number of trees on a tree farm. **(p. 685)**
Use sequences and series to solve real-life problems. **(11.1–11.5)**	Find the total length of the vertical supports used to build a roof. **(p. 656)**

How does Chapter 11 fit into the BIGGER PICTURE of algebra?

Since elementary school you have studied number patterns (sequences). Now you can use algebra to write and use rules for sequences and series. An arithmetic sequence has a common difference, so it is similar to a linear function. A geometric sequence has a common ratio, so it is similar to an exponential function. Recursive rules are used in computer programs and in spreadsheet formulas.

STUDY STRATEGY

How did you learn by teaching?

Here is an example of an explanation given by one student to another, following the **Study Strategy** on page 650.

Learn by Teaching

Write a rule for the nth term of this sequence:

$2, 7, 12, 17, \ldots$

"First look for a common difference or a common ratio. $7 - 2 = 5$, $12 - 7 = 5$, and $17 - 12 = 5$, so the common difference is 5. So $5n$ will be part of the rule. If $n = 1$, $5n = 5$. But the first term is 2, which is 3 less than 5, so we need to subtract 3. Let's try $5n - 3$ and see if it works. $5(1) - 3 = 2$, $5(2) - 3 = 7$, $5(3) - 3 = 12$, and $5(4) - 3 = 17$. Yes, it works. So the rule is $a_n = 5n - 3$."

CHAPTER 11

Chapter Review

VOCABULARY

- terms of a sequence, p. 651
- sequence, p. 651
- finite sequence, p. 651
- infinite sequence, p. 651
- series, p. 653
- summation notation, p. 653
- sigma notation, p. 653
- arithmetic sequence, p. 659
- common difference, p. 659
- arithmetic series, p. 661
- geometric sequence, p. 666
- common ratio, p. 666
- geometric series, p. 668
- explicit rule, p. 681
- recursive rule, p. 681
- factorial, p. 681

11.1 An Introduction to Sequences and Series

Examples on pp. 651–654

EXAMPLES You can find the first four terms of the sequence $a_n = 3n - 7$.

$a_1 = 3(1) - 7 = -4$ ← first term

$a_2 = 3(2) - 7 = -1$ ← second term

$a_3 = 3(3) - 7 = 2$ ← third term

$a_4 = 3(4) - 7 = 5$ ← fourth term

The *sequence* defined by $a_n = 3n - 7$ is $-4, -1, 2, 5, \ldots$.

The associated *series* is the sum of the terms of the sequence: $(-4) + (-1) + 2 + 5 + \cdots$.

You can use summation notation to write the series $2 + 4 + 6 + 8 + 10$ as $\sum_{i=1}^{5} 2i$.

You can find the sum of a series by adding the terms or by using formulas for special series.

The sum of the series $\sum_{i=1}^{22} i^2$ is $\frac{n(n+1)(2n+1)}{6} = \frac{22(22+1)(2(22)+1)}{6} = 3795$.

Write the first six terms of the sequence.

1. $a_n = n^2 + 5$ **2.** $a_n = (n+1)^3$ **3.** $a_n = 6 - 2n$ **4.** $a_n = \frac{n}{n+3}$

Write the next term in the sequence. Then write a formula for the *n*th term.

5. $2, 4, 6, 8, \ldots$ **6.** $-3, 6, -12, 24, \ldots$ **7.** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

Write the series with summation notation.

8. $4 + 8 + 12 + 16$ **9.** $1 + 2 + 3 + 4 + \cdots$ **10.** $0 + 3 + 6 + 9 + 12$

Find the sum of the series.

11. $\sum_{n=1}^{25} n^2$ **12.** $\sum_{n=4}^{10} n(2n-1)$ **13.** $\sum_{i=1}^{12} i$ **14.** $\sum_{k=1}^{30} 4$

11.2 Arithmetic Sequences and Series

Examples on pp. 659–662

EXAMPLES The sequence 4, 7, 10, 13, 16, . . . is an arithmetic sequence because the difference between consecutive terms is constant:

$7 - 4 = 3 \qquad 10 - 7 = 3 \qquad 13 - 10 = 3 \qquad 16 - 13 = 3$

The common difference is 3, so $d = 3$.

A rule for the nth term of this arithmetic sequence is:

$$a_n = a_1 + (n - 1)d = 4 + (n - 1)3 = 3n + 1$$

The sum of the first 20 terms of this arithmetic series is:

$$S_{20} = 20\left(\frac{a_1 + a_{20}}{2}\right) = 20\left(\frac{4 + 61}{2}\right) = 650$$

Write a rule for the *n*th term of the arithmetic sequence.

15. 1, 7, 13, 19, 25, . . .
16. 4, 6, 8, 10, 12, . . .
17. 3.5, 3, 2.5, 2, 1.5, . . .
18. $d = 5, a_1 = 13$
19. $d = -2, a_9 = 3$
20. $a_4 = 20, a_{13} = 65$

Find the sum of the first *n* terms of the arithmetic series.

21. $8 + 20 + 32 + 44 + \cdots; n = 14$
22. $(-6) + (-2) + 2 + 6 + \cdots; n = 20$
23. $0.5 + 0.9 + 1.3 + 1.7 + \cdots; n = 54$
24. $(-12) + (-8) + (-4) + 0 + \cdots; n = 40$

11.3 Geometric Sequences and Series

Examples on pp. 666–669

EXAMPLES The sequence 5, 15, 45, 135, 405, . . . is a geometric sequence because the ratio of any term to the previous term is constant:

$\frac{15}{5} = 3 \qquad \frac{45}{15} = 3 \qquad \frac{135}{45} = 3 \qquad \frac{405}{135} = 3$

The common ratio is 3, so $r = 3$.

A rule for the nth term of this geometric sequence is:

$$a_n = a_1r^{n-1} = 5(3)^{n-1}$$

The sum of the first 8 terms of this geometric series is:

$$S_8 = a_1\left(\frac{1 - r^8}{1 - r}\right) = 5\left(\frac{1 - 3^8}{1 - 3}\right) = 16{,}400$$

Write a rule for the *n*th term of the geometric sequence.

25. 64, 32, 16, 8, 4, . . .
26. 6, 12, 24, 48, . . .
27. 200, 20, 2, 0.2, 0.02, . . .
28. $r = 3, a_1 = 6$
29. $r = -\frac{1}{4}, a_4 = 1$
30. $a_2 = 50, a_6 = 0.005$

Find the sum of the series.

31. $\sum_{i=1}^{5} 16(2)^{i-1}$
32. $\sum_{i=1}^{10} 20(0.2)^{i-1}$
33. $\sum_{i=0}^{6} 10\left(\frac{1}{2}\right)^i$
34. $\sum_{i=1}^{8} 2\left(\frac{3}{5}\right)^{i-1}$

11.4 INFINITE GEOMETRIC SERIES

Examples on pp. 675–677

EXAMPLES You can find the sum of the infinite geometric series

$\sum_{n=1}^{\infty} 4\left(\frac{3}{5}\right)^{n-1}$ because $|r| = \left|\frac{3}{5}\right| < 1$: $S = \frac{a_1}{1-r} = \frac{4}{1-\frac{3}{5}} = 10$.

The infinite geometric series $\sum_{n=1}^{\infty} \frac{1}{2}(5)^{n-1}$ has no sum because $|r| = |5| \geq 1$.

Find the sum of the infinite geometric series.

35. $\sum_{n=1}^{\infty} 15\left(\frac{2}{9}\right)^{n-1}$ **36.** $\sum_{n=1}^{\infty} 3\left(\frac{3}{4}\right)^{n-1}$ **37.** $\sum_{n=1}^{\infty} 5(0.8)^{n-1}$ **38.** $\sum_{n=1}^{\infty} 4(-0.2)^{n-1}$

Find the common ratio of the infinite geometric series with the given sum and first term.

39. $S = 18, a_1 = 12$ **40.** $S = 2, a_1 = 0.5$ **41.** $S = 20, a_1 = 4$ **42.** $S = -5, a_1 = -2$

43. $S = -10, a_1 = -3$ **44.** $S = 6, a_1 = \frac{1}{3}$ **45.** $S = \frac{1}{4}, a_1 = \frac{1}{16}$ **46.** $S = 3\frac{1}{3}, a_1 = 6$

Write the repeating decimal as a fraction.

47. 0.222 . . . **48.** 0.4545 . . . **49.** 39.3939 . . . **50.** 0.001001 . . .

11.5 RECURSIVE RULES FOR SEQUENCES

Examples on pp. 681–683

EXAMPLES You can find the first five terms of the sequence defined by the recursive rule $a_1 = 3, a_n = a_{n-1} + n + 6$.

$a_1 = 3$ ← first term

$a_2 = a_{n-1} + n + 6 = a_1 + 2 + 6 = 3 + 2 + 6 = 11$ ← second term

$a_3 = a_{n-1} + n + 6 = a_2 + 3 + 6 = 11 + 3 + 6 = 20$ ← third term

$a_4 = a_{n-1} + n + 6 = a_3 + 4 + 6 = 20 + 4 + 6 = 30$ ← fourth term

$a_5 = a_{n-1} + n + 6 = a_4 + 5 + 6 = 30 + 5 + 6 = 41$ ← fifth term

The sequence is 3, 11, 20, 30, 41, ...

A recursive formula for the sequence 1, 5, 14, 30, . . . is $a_1 = 1, a_n = a_{n-1} + n^2$.

Write the first six terms of the sequence.

51. $a_1 = 10$, $a_n = 4a_{n-1}$ **52.** $a_1 = 1$, $a_n = n \cdot a_{n-1}$ **53.** $a_1 = 2$, $a_n = a_{n-1} - n$ **54.** $a_1 = -1$, $a_n = (a_{n-1})^2 + 3$

Write a recursive rule for the sequence. The sequence may be arithmetic, geometric, or neither.

55. 7, 14, 28, 56, 112, . . . **56.** 4, 8, 13, 19, 26, . . . **57.** 1, 6, 11, 16, 21, . . .

58. 200, 100, 50, 25, . . . **59.** 1, 2, 5, 26, 677, . . . **60.** −2, −6, −12, −20, . . .

CHAPTER 11

Chapter Test

Tell whether the sequence is *arithmetic*, *geometric*, or *neither*. Explain your answer.

1. $-5, -3, -1, 1, \ldots$

2. $-4, -2, 2, 4, \ldots$

3. $12, 6, 3, \frac{3}{2}, \ldots$

4. $\frac{1}{3}, 1, 3, 9, \ldots$

Write the first six terms of the sequence.

5. $a_n = n^2 + 1$

6. $a_n = 3n - 5$

7. $a_1 = 4$
$a_n = n + a_{n-1}$

8. $a_1 = 1$
$a_n = 2a_{n-1}$

Write the next term of the sequence, and then write a rule for the *n*th term.

9. $2, 4, 8, 16, \ldots$

10. $4, 9, 14, 19, \ldots$

11. $2, 10, 50, 250, \ldots$

12. $-9, -10, -11, -12, \ldots$

13. $5, -\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \ldots$

14. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

15. $\frac{3}{2}, \frac{4}{4}, \frac{5}{6}, \frac{6}{8}, \ldots$

16. $1.1, 2.2, 3.3, 4.4, \ldots$

Write a recursive rule for the sequence. (Recall that *d* is the common difference of an arithmetic sequence and *r* is the common ratio of a geometric sequence.)

17. $r = 0.3, a_1 = 4$

18. $d = 4, a_1 = 1$

19. $40, 20, 10, 5, \ldots$

20. $2, 8, 18, 32, 50, \ldots$

Find the sum of the series.

21. $\sum_{i=1}^{100} i$

22. $\sum_{i=2}^{5} \frac{1}{2}i^2$

23. $\sum_{i=1}^{6} (i - 10)$

24. $\sum_{i=1}^{20} (3i + 2)$

25. $\sum_{i=1}^{5} 7(-2)^{i-1}$

26. $\sum_{i=0}^{9} 5\left(\frac{1}{4}\right)^i$

27. $\sum_{i=1}^{\infty} 64\left(-\frac{1}{2}\right)^{i-1}$

28. $\sum_{i=1}^{\infty} 100\left(\frac{7}{10}\right)^{i-1}$

29. Find the sum of the first 30 terms of the arithmetic sequence $3, 7, 11, 15, \ldots$.

30. Find the sum of the infinite geometric series $2 + 1 + 0.5 + 0.25 + \cdots$.

31. Write the series $1 + 3 + 5 + 7 + 9 + 11$ with summation notation.

32. Write the repeating decimal $0.7575\ldots$ as a fraction.

33. **FALLING OBJECT** An object is dropped from an airplane. During the first second, the object falls 4.9 meters. During the second second, it falls 14.7 meters. During the third second, it falls 24.5 meters. During the fourth second, it falls 34.3 meters. If this pattern continues, how far will the object fall during the tenth second? Find the total distance the object will fall after 10 seconds.

34. **CELL DIVISION** In early growth of an embryo, a human cell divides into two cells, each of which divides into two cells, and so on. The number a_n of new cells formed after the nth division is $a_n = 2^{n-1}$. Find the sum of the first 9 terms of the series to find the total number of new cells after the 8th division.

35. **SPRING** The length of the first loop of a spring is 20 inches. The length of the second loop is $\frac{9}{10}$ of the length of the first loop. The length of the third loop is $\frac{9}{10}$ of the length of the second loop, and so on. If the spring could have infinitely many loops, would its length be finite? If so, find the length.