CHAPTER 11

Extension

What you should learn

GOAL Use mathematical induction to prove statements about all positive integers.

Mathematical Induction

In Lesson 11.1 you saw that the rule for the sum of the first n positive integers is:

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Statements like the one above about all positive integers can be proven using a method called *mathematical induction*.

MATHEMATICAL INDUCTION

To show that a statement is true for all positive integers n, perform these steps.

BASIS STEP: Show that the statement is true for $n = 1$.

INDUCTIVE STEP: Assume that the statement is true for $n = k$ where k is any positive integer. Show that this implies the statement is true for $n = k + 1$.

Mathematical induction works as follows. If you know from the basis step that a statement is true for $n = 1$, then the inductive step implies that it is true for $n = 2$, and therefore, for $n = 3$, and so on for all positive integers n.

EXAMPLE 1 *Using Mathematical Induction*

Use mathematical induction to prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

SOLUTION

Basis step: Check to see if the formula works for $n = 1$.

$$1 \stackrel{?}{=} \frac{(1)(1+1)}{2} \longrightarrow 1 = 1\ \checkmark$$

Inductive step: Assume that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.

Show that $1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$.

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2} \qquad \text{Assume true for } k.$$

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) \qquad \text{Add } k+1 \text{ to each side.}$$

$$= \frac{k(k+1) + 2(k+1)}{2} \qquad \text{Add.}$$

$$= \frac{(k+1)(k+2)}{2} \qquad \text{Factor out } k+1.$$

$$= \frac{(k+1)[(k+1)+1]}{2} \qquad \text{Rewrite } k+2 \text{ as } (k+1)+1.$$

Therefore, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all positive integers n.

Florida Standards and Assessment
MA.D.1.4.1

EXAMPLE 2 *Using Mathematical Induction*

Let $a_n = 3a_{n-1} + 1$ with $a_1 = 1$. Use mathematical induction to prove that an explicit rule for the nth term is $a_n = \frac{3^n - 1}{2}$.

Solution

Basis step: Check to see that the rule works for $n = 1$.

$$a_1 \stackrel{?}{=} \frac{3^1 - 1}{2} \longrightarrow 1 = 1 \checkmark$$

Inductive step: Assume that $a_k = \frac{3^k - 1}{2}$. Show that $a_{k+1} = \frac{3^{k+1} - 1}{2}$.

$$a_{k+1} = 3a_k + 1 \quad \text{Definition of } a_n \text{ for } n = k + 1$$

$$= 3\left(\frac{3^k - 1}{2}\right) + 1 \quad \text{Substitute for } a_k.$$

$$= \frac{3^{k+1} - 3}{2} + 1 \quad \text{Multiply.}$$

$$= \frac{3^{k+1} - 3 + 2}{2} \quad \text{Add.}$$

$$= \frac{3^{k+1} - 1}{2} \quad \text{Simplify.}$$

Therefore, an explicit rule for the nth term is $a_n = \frac{3^n - 1}{2}$ for all positive integers n.

EXERCISES

Use mathematical induction to prove the statement.

1. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

2. $\sum_{i=1}^{n} (2i + 1) = n(n + 2)$

3. $\sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right)$

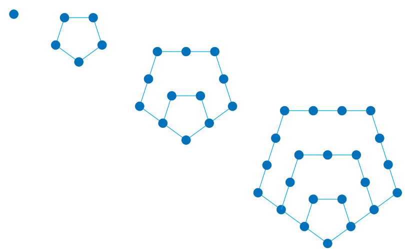
4. $\sum_{i=1}^{n} (2i)^2 = \frac{2n(n+1)(2n+1)}{3}$

5. $\sum_{i=1}^{n} 5^i = \frac{5^{n+1} - 5}{4}$

6. $\sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = 1 - \left(\frac{1}{2}\right)^n$

7. **GEOMETRY CONNECTION** The numbers 1, 5, 12, 22, 35, 51, . . . are called *pentagonal numbers* because they represent the numbers of dots used to make pentagons, as shown at the right. Prove that the nth pentagonal number P_n is given by:

$$P_n = \frac{n(3n - 1)}{2}$$



8. **LOGICAL REASONING** Let $f_1, f_2, \ldots, f_n, \ldots$ be the Fibonacci sequence. Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.