11.4 Infinite Geometric Series

What you should learn

GOAL 1 Find sums of infinite geometric series.

GOAL 2 Use infinite geometric series as models of **real-life** situations, such as the distance traveled by a bouncing ball in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the spending generated by tourists in Malaysia in **Exs. 50 and 51**.

REAL LIFE

GOAL 1 Using Infinite Geometric Series

Consider the following infinite geometric series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Even though this series has infinitely many terms, it has a finite sum! To see this, compute and graph the sum of the first n terms for several values of n.

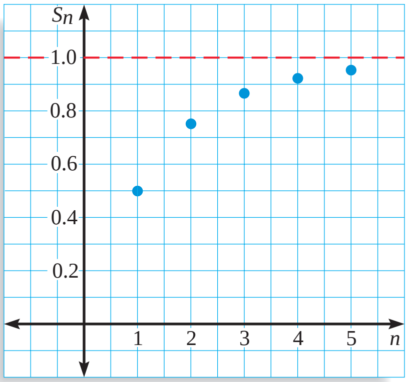
$$S_1 = \frac{1}{2} = 0.5$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = 0.75$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \approx 0.88$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \approx 0.94$$

$$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \approx 0.97$$



Notice that S_n appears to be approaching 1 as n increases. To see why this makes sense, consider the rule for S_n:

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right) = \frac{1}{2}\left(\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\right) = 1 - \left(\frac{1}{2}\right)^n$$

As n increases, $\left(\frac{1}{2}\right)^n$ gets closer and closer to 0, which means that S_n gets closer and closer to 1. The same is true for r^n provided r is between -1 and 1. Therefore, the formula for the sum of a *finite* geometric series, $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$, approaches the formula below as n increases.

THE SUM OF AN INFINITE GEOMETRIC SERIES

The sum of an infinite geometric series with first term a_1 and common ratio r is given by

$$S = \frac{a_1}{1 - r}$$

provided $|r| < 1$. If $|r| \geq 1$, the series has no sum.

For the series described above, the sum is $S = \dfrac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$, as expected.

EXAMPLE 1 *Finding Sums of Infinite Geometric Series*

Find the sum of the infinite geometric series.

a. $\sum_{i=1}^{\infty} 3(0.7)^{i-1}$

b. $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots$

SOLUTION

a. For this series, $a_1 = 3$ and $r = 0.7$.

$$S = \frac{a_1}{1-r} = \frac{3}{1-0.7} = 10$$

b. For this series, $a_1 = 1$ and $r = -\frac{1}{4}$.

$$S = \frac{a_1}{1-r} = \frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{4}{5}$$

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

EXAMPLE 2 *Finding the Common Ratio*

An infinite geometric series with first term $a_1 = 4$ has a sum of 10. What is the common ratio of the series?

SOLUTION

$S = \frac{a_1}{1-r}$ — **Write rule for sum.**

$10 = \frac{4}{1-r}$ — **Substitute for S and a_1.**

$10(1-r) = 4$ — **Multiply each side by $1 - r$.**

$1 - r = \frac{2}{5}$ — **Divide each side by 10.**

$r = \frac{3}{5}$ — **Solve for r.**

▶ The common ratio is $r = \frac{3}{5}$.

EXAMPLE 3 *Writing a Repeating Decimal as a Fraction*

Write 0.181818 . . . as a fraction.

SOLUTION

$0.181818\ldots = 18(0.01) + 18(0.01)^2 + 18(0.01)^3 + \cdots$

$= \frac{a_1}{1-r}$ — **Write rule for sum.**

$= \frac{18(0.01)}{1-0.01}$ — **Substitute for a_1 and r.**

$= \frac{18}{99}$ — **Write as a quotient of integers.**

$= \frac{2}{11}$ — **Simplify.**

▶ The repeating decimal 0.181818 . . . is $\frac{2}{11}$ as a fraction.

STUDENT HELP

Study Tip
You can check the result in Example 3 by dividing 2 by 11 on a calculator.

FOCUS ON APPLICATIONS

BALL BOUNCE This photo of a ball bouncing was taken with time-lapse photography. The images of the ball get closer together as you move up, which means the ball's speed is decreasing.

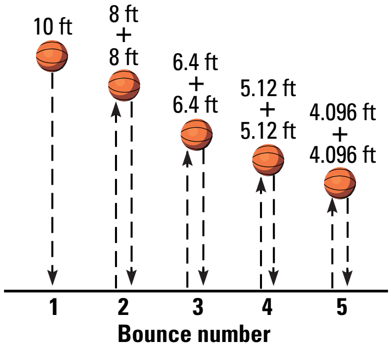
GOAL 2 INFINITE GEOMETRIC SERIES IN REAL LIFE

EXAMPLE 4 *Using an Infinite Series as a Model*

BALL BOUNCE A ball is dropped from a height of 10 feet. Each time it hits the ground, it bounces to 80% of its previous height.

a. Find the total distance traveled by the ball.

b. On which bounce will the ball have traveled 85% of its total distance?



SOLUTION

a. The total distance traveled by the ball is:

$$d = \underbrace{10}_{\text{down}} + \underbrace{10(0.8)}_{\text{up}} + \underbrace{10(0.8)}_{\text{down}} + \underbrace{10(0.8)^2}_{\text{up}} + \underbrace{10(0.8)^2}_{\text{down}} + \underbrace{10(0.8)^3}_{\text{up}} + \cdots$$

$$= 10 + 2[10(0.8)] + 2[10(0.8)^2] + 2[10(0.8)^3] + \cdots$$

$$= 10 + 20(0.8) + 20(0.8)^2 + 20(0.8)^3 + \cdots$$

$$= 10 + \frac{20(0.8)}{1 - 0.8}$$ **Excluding first term, find sum of series.**

$$= 10 + 80$$ **Simplify fraction.**

$$= 90$$ **Simplify.**

▶ The ball travels a total distance of 90 feet.

b. Let n be the number of up-and-down bounces. The distance d_n the ball travels is:

$$d_n = \underbrace{10}_{\text{down-only distance}} + \underbrace{20(0.8)\left(\frac{1 - (0.8)^n}{1 - 0.8}\right)}_{\text{sum of } n \text{ up-and-down bounces}}$$ **Write rule for d_n.**

$$0.85(90) = 10 + 20(0.8)\left(\frac{1 - (0.8)^n}{1 - 0.8}\right)$$ **Substitute for d_n.**

$$76.5 = 10 + 16\left(\frac{1 - (0.8)^n}{1 - 0.8}\right)$$ **Simplify.**

$$4.156 \approx \frac{1 - (0.8)^n}{0.2}$$ **Isolate fraction.**

$$0.831 \approx 1 - (0.8)^n$$ **Multiply each side by 0.2.**

$$(0.8)^n \approx 0.169$$ **Isolate exponential expression.**

$$n \approx \frac{\log 0.169}{\log 0.8} \approx 7.97$$ **Solve for n.**

▶ The ball travels 85% of its total distance after about 8 up-and-down bounces, or after 9 bounces including the first down-only bounce.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: A(n) __?__ geometric series has infinitely many terms.

Concept Check ✓

2. Under what conditions will $\sum_{i=1}^{\infty} a_1 r^{i-1}$ have a sum?

3. What two things do you need to know to find the sum of an infinite geometric series?

Skill Check ✓

Find the sum of the infinite geometric series.

4. $\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1}$ **5.** $-2 + \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \cdots$

Find the common ratio of the infinite geometric series with the given sum and first term.

6. $S = 6,\ a_1 = 1$ **7.** $S = 12,\ a_1 = 2$ **8.** $S = 10\frac{1}{2},\ a_1 = \frac{1}{2}$

Write the repeating decimal as a fraction.

9. 0.555 . . . **10.** 0.1212 . . . **11.** 245.245245 . . .

12. **BALL BOUNCE** A ball is dropped from a height of 5 feet. Each time it hits the ground, it bounces one half of its previous height.

a. Find the total distance traveled by the ball.

b. On which bounce will the ball have traveled 75% of its total distance?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 956.

IDENTIFYING A SUM **Decide whether the infinite geometric series has a sum. Explain why or why not.**

13. $\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n-1}$ **14.** $\sum_{n=0}^{\infty} -5\left(\frac{1}{5}\right)^{n}$ **15.** $\sum_{n=1}^{\infty} \frac{3}{2}\left(\frac{1}{3}\right)^{n-1}$ **16.** $\sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{4}{3}\right)^{n}$

FINDING SUMS **Find the sum of the infinite geometric series if it has one.**

17. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n}$ **18.** $\sum_{n=0}^{\infty} 3\left(\frac{2}{3}\right)^{n}$ **19.** $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1}$ **20.** $\sum_{n=0}^{\infty} \frac{2}{7}(2)^{n}$

21. $\sum_{n=0}^{\infty} 4\left(\frac{1}{4}\right)^{n}$ **22.** $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^{n-1}$ **23.** $\sum_{n=0}^{\infty} 2\left(\frac{6}{5}\right)^{n}$ **24.** $\sum_{n=0}^{\infty} 4\left(\frac{3}{7}\right)^{n}$

25. $\sum_{n=0}^{\infty} -\frac{1}{8}\left(-\frac{1}{2}\right)^{n}$ **26.** $\sum_{n=1}^{\infty} \frac{1}{2}\left(-\frac{2}{5}\right)^{n-1}$ **27.** $\sum_{n=0}^{\infty} \frac{1}{12}\left(-\frac{3}{25}\right)^{n}$ **28.** $\sum_{n=1}^{\infty} -\left(-\frac{2}{11}\right)^{n-1}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 13–28
Example 2: Exs. 29–37
Example 3: Exs. 38–46
Example 4: Exs. 47–51

FINDING COMMON RATIOS **Find the common ratio of the infinite geometric series with the given sum and first term.**

29. $S = 4,\ a_1 = 1$ **30.** $S = 10,\ a_1 = 1$ **31.** $S = 12,\ a_1 = 3$

32. $S = 8,\ a_1 = 2$ **33.** $S = 6,\ a_1 = 2$ **34.** $S = 50,\ a_1 = 4$

35. $S = -\frac{1}{9},\ a_1 = -\frac{1}{6}$ **36.** $S = -\frac{11}{13},\ a_1 = -1$ **37.** $S = 2\frac{2}{9},\ a_1 = 4$

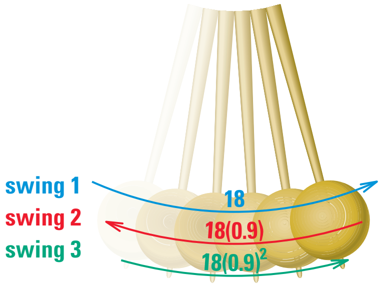
WRITING REPEATING DECIMALS **Write the repeating decimal as a fraction.**

38. 0.444 . . . **39.** 0.777 . . . **40.** 0.999 . . .

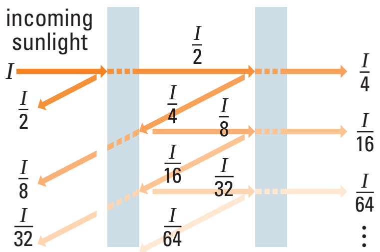
41. 0.5151 . . . **42.** 0.2323 . . . **43.** 0.1616 . . .

44. 63.6363 . . . **45.** 120.120120 . . . **46.** 297.297297 . . .

47. **PENDULUM** A pendulum is released to swing freely. On the first swing, the pendulum travels a distance of 18 inches. On each successive swing, the pendulum travels 90% of the distance of the previous swing. What is the total distance the pendulum swings? After how many swings has the pendulum traveled 80% of its total distance?



48. **WINDOWS** Some types of windows are constructed with two parallel panes of glass, each of which reflects half of the sunlight that hits it from either side. The other half of the sunlight passes through the pane. How much of the sunlight will pass through *both* panes?



49. **ZENO'S PARADOX** Can the Greek hero Achilles, running at 20 feet per second, ever catch a tortoise, starting 20 feet away and running at 10 feet per second? The Greek mathematician Zeno said no. He reasoned as follows:

- When Achilles runs 20 feet the tortoise will be in a new spot, 10 feet away.
- Then, when Achilles gets to that spot, the tortoise will be 5 feet away.
- Achilles will keep cutting the distance in half but will never catch the tortoise.

In actuality, looking at the race as Zeno did, you can see that both the distances and the times required to achieve them form infinite geometric series. Using the table, show that both series have finite sums. What do these sums represent?

Distance (ft)	20	10	5	2.5	1.25	0.625	. . .
Time (sec)	1	0.5	0.25	0.125	0.0625	0.03125	. . .

FOCUS ON CAREERS

REAL LIFE **ECONOMIST** Economists study the ways a society distributes resources to produce goods and services. They conduct research, monitor economic trends, and make projections.

INTERNET **CAREER LINK**
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TOURISM **In Exercises 50 and 51, use the following information.**
In 1974 the Malaysian Tourist Development Corporation studied the economic impact of distributing tourist brochures. It was estimated that M$4.72 ("M$" means Malaysian dollars) in additional money was spent by tourists for every brochure distributed in the capital city of Kuala Lumpur. It was also estimated that for each M$1 spent on goods or services, 80.5% of that would be re-spent, creating a "multiplier" effect. (That is, each Malaysian dollar spent would lead to total spending of M$1 + (0.805)M$1 + (0.805)(0.805)M$1 + (0.805)(0.805)(0.805)M$1 + · · · .)

50. What total spending was generated by a tourist spending M$1 in 1974?

51. How much total spending would be generated by the average tourist who received a brochure?

52. **LOGICAL REASONING** Find two different infinite geometric series whose sum is 3.

Test Preparation

53. MULTIPLE CHOICE An infinite geometric series with first term $a_1 = 24$ has a sum of 48. What is the common ratio of the series?

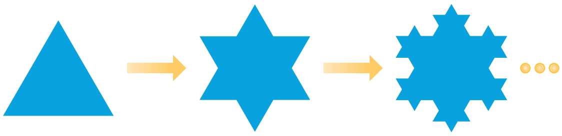
(A) $r = 1$ (B) $r = \frac{1}{2}$ (C) $r = 2$ (D) $r = \frac{1}{24}$ (E) $r = \frac{3}{8}$

54. MULTIPLE CHOICE The repeating decimal 18.181818 . . . is equivalent to what fraction?

(A) $\frac{891}{50}$ (B) $\frac{2}{11}$ (C) $\frac{181}{9}$ (D) $\frac{200}{11}$ (E) $\frac{1783}{99}$

★ Challenge

55. GEOMETRY CONNECTION A *Koch snowflake* is created by starting with an equilateral triangle with sides 1 unit long. Then, on the middle third of each side of the triangle, a new equilateral triangle is constructed. This process is repeated as shown.



a. Recall that the area of an equilateral triangle with side length s can be found using the formula $A = \frac{s^2\sqrt{3}}{4}$. Use this formula to complete the table below.

	Figure 1	Figure 2	Figure 3	Figure 4	. . .	Figure n
Number of new triangles	1	3	?	?	. . .	?
Area of each new triangle	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{36}$	?	?	. . .	?
Total new area	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{12}$	?	?	. . .	?

b. What is the total area of the Koch snowflake? (*Hint:* Add up the entries in the last row of the table.)

EXTRA CHALLENGE
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MIXED REVIEW

IDENTIFYING ASYMPTOTES Identify the horizontal and vertical asymptotes. State the domain and range of the function. (Review 9.2)

56. $f(x) = \frac{5}{x}$ **57.** $f(x) = -\frac{5}{x}$ **58.** $f(x) = \frac{7}{x} - 3$

59. $f(x) = \frac{x - 5}{x + 7}$ **60.** $f(x) = \frac{4x + 3}{-2x + 17}$ **61.** $f(x) = \frac{2.2x + 3.1}{x - 0.7}$

GRAPHING Graph the equation. (Review 10.4)

62. $\frac{x^2}{4} + \frac{y^2}{16} = 1$ **63.** $\frac{25x^2}{4} + \frac{25y^2}{9} = 1$ **64.** $\frac{x^2}{16} + y^2 = 1$

WRITING RULES Write a rule for the nth term of the sequence. Recall that d is the common difference of an arithmetic sequence and r is the common ratio of a geometric sequence. (Review 11.2, 11.3 for 11.5)

65. $d = 4, a_1 = 1$ **66.** $d = 8, a_1 = -2$ **67.** $d = -2, a_2 = 9$

68. $r = 2, a_3 = -20$ **69.** $r = 0.5, a_1 = 4$ **70.** $r = 0.875, a_1 = -10$