11.2 Arithmetic Sequences and Series

What you should learn

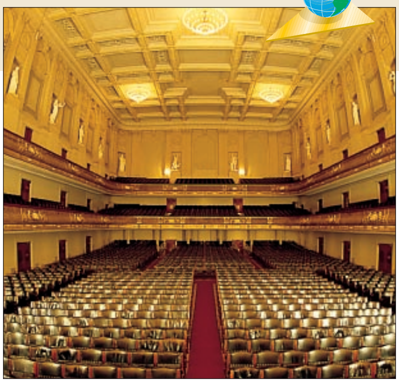
GOAL 1 Write rules for arithmetic sequences and find sums of arithmetic series.

GOAL 2 Use arithmetic sequences and series in **real-life** problems, such as finding the number of cells in a honeycomb in **Ex. 57**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the number of seats in a concert hall in **Example 7**.

REAL LIFE



GOAL 1 USING ARITHMETIC SEQUENCES AND SERIES

In an **arithmetic sequence**, the difference between consecutive terms is constant. The constant difference is called the **common difference** and is denoted by d.

EXAMPLE 1 *Identifying Arithmetic Sequences*

Decide whether each sequence is arithmetic.

a. $-3, 1, 5, 9, 13, \ldots$

b. $2, 5, 10, 17, 26, \ldots$

SOLUTION

To decide whether a sequence is arithmetic, find the differences of consecutive terms.

a. $a_2 - a_1 = 1 - (-3) = 4$

$a_3 - a_2 = 5 - 1 = 4$

$a_4 - a_3 = 9 - 5 = 4$

$a_5 - a_4 = 13 - 9 = 4$

Each difference is 4, so the sequence is arithmetic.

b. $a_2 - a_1 = 5 - 2 = 3$

$a_3 - a_2 = 10 - 5 = 5$

$a_4 - a_3 = 17 - 10 = 7$

$a_5 - a_4 = 26 - 17 = 9$

The differences are not constant, so the sequence is not arithmetic.

RULE FOR AN ARITHMETIC SEQUENCE

The nth term of an arithmetic sequence with first term a_1 and common difference d is given by:

$$a_n = a_1 + (n - 1)d$$

EXAMPLE 2 *Writing a Rule for the nth Term*

Write a rule for the nth term of the sequence 50, 44, 38, 32, ... Then find a_{20}.

SOLUTION

The sequence is arithmetic with first term $a_1 = 50$ and common difference $d = 44 - 50 = -6$. So, a rule for the nth term is:

$a_n = a_1 + (n - 1)d$ — **Write general rule.**

$= 50 + (n - 1)(-6)$ — **Substitute for a_1 and d.**

$= 56 - 6n$ — **Simplify.**

The 20th term is $a_{20} = 56 - 6(20) = -64$.

EXAMPLE 3 Finding the nth Term Given a Term and the Common Difference

One term of an arithmetic sequence is $a_{13} = 30$. The common difference is $d = \frac{3}{2}$.

a. Write a rule for the nth term. **b.** Graph the sequence.

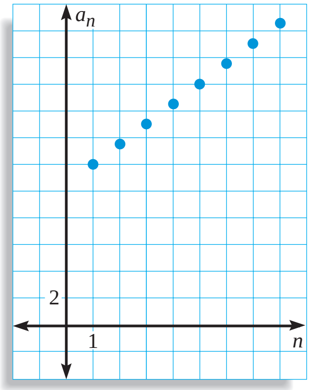
SOLUTION

a. Begin by finding the first term as follows.

$a_n = a_1 + (n - 1)d$	Write rule for *n*th term.
$a_{13} = a_1 + (13 - 1)d$	Substitute 13 for *n*.
$30 = a_1 + 12\left(\frac{3}{2}\right)$	Substitute for a_{13} and *d*.
$12 = a_1$	Solve for a_1.

So, a rule for the nth term is:

$a_n = a_1 + (n - 1)d$	Write general rule.
$= 12 + (n - 1)\frac{3}{2}$	Substitute for a_1 and *d*.
$= \frac{21}{2} + \frac{3}{2}n$	Simplify.



b. The graph is shown at the right. Notice that the points lie on a line. This is true for *any* arithmetic sequence.

EXAMPLE 4 Finding the nth Term Given Two Terms

Two terms of an arithmetic sequence are $a_6 = 10$ and $a_{21} = 55$.

a. Find a rule for the nth term. **b.** Find the value of n for which $a_n = 40$.

SOLUTION

a. ***Write*** a system of equations using $a_n = a_1 + (n - 1)d$ and substituting 21 for n (Equation 1) and then 6 for n (Equation 2).

$a_{21} = a_1 + (21 - 1)d$ →	$55 = a_1 + 20d$	Equation 1
$a_6 = a_1 + (6 - 1)d$ →	$10 = a_1 + 5d$	Equation 2
Solve the system.	$45 = 15d$	Subtract equations.
	$3 = d$	Solve for *d*.
	$55 = a_1 + 20(3)$	Substitute for *d*.
	$-5 = a_1$	Solve for a_1.

Find a rule for a_n.	$a_n = a_1 + (n - 1)d$	Write general rule.
	$a_n = -5 + (n - 1)3$	Substitute for a_1 and *d*.
	$a_n = -8 + 3n$	Simplify.

b. $a_n = -8 + 3n$	Use the rule for a_n from part (a).
$40 = -8 + 3n$	Substitute 40 for a_n.
$16 = n$	Solve for *n*.

FOCUS ON PEOPLE

KARL FRIEDRICH GAUSS, a famous nineteenth century mathematician, was a child prodigy. It is said that when Gauss was ten his teacher asked his class to add the numbers from 1 to 100. Almost immediately Gauss found the answer by mentally figuring the summation.

The expression formed by adding the terms of an arithmetic sequence is called an **arithmetic series**. The sum of the first n terms of an arithmetic series is denoted by S_n. To find a rule for S_n, you can write S_n in two different ways and add the results.

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + a_n \\ S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + a_1 \\ \hline 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) \end{aligned}$$

You can conclude that $2S_n = n(a_1 + a_n)$, which leads to the following result.

THE SUM OF A FINITE ARITHMETIC SERIES

The sum of the first n terms of an arithmetic series is:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

In words, S_n is the mean of the first and nth terms, multiplied by the number of terms.

EXAMPLE 5 *Finding a Sum*

Consider the arithmetic series $4 + 7 + 10 + 13 + 16 + 19 + \cdots$.

a. Find the sum of the first 30 terms. **b.** Find n such that $S_n = 175$.

SOLUTION

a. To begin, notice that $a_1 = 4$ and $d = 3$. So, a formula for the nth term is:

$a_n = a_1 + (n - 1)d$ — Write rule for the nth term.

$= 4 + (n - 1)3$ — Substitute for a_1 and d.

$= 1 + 3n$ — Simplify.

The 30th term is $a_{30} = 1 + 3(30) = 91$. So, the sum of the first 30 terms is:

$S_{30} = 30\left(\frac{a_1 + a_{30}}{2}\right)$ — Write rule for S_{30}.

$= 30\left(\frac{4 + 91}{2}\right)$ — Substitute for a_1 and a_{30}.

$= 1425$ — Simplify.

▶ The sum of the first 30 terms is 1425.

b. $n\left(\frac{4 + (1 + 3n)}{2}\right) = 175$ — Use rule for S_n.

$5n + 3n^2 = 350$ — Multiply each side by 2.

$3n^2 + 5n - 350 = 0$ — Write in standard form.

$(3n + 35)(n - 10) = 0$ — Factor.

$n = 10$ — Choose positive solution.

▶ So, $S_n = 175$ when $n = 10$.

STUDENT HELP

Look Back
For help with quadratic equations, see p. 256.

FOCUS ON APPLICATIONS

PHILHARMONIE HALL in Berlin has 2335 seats. This hall was the first to use an architectural concept called "vineyard" design to reflect sound to the audience using several audience tiers at different heights.

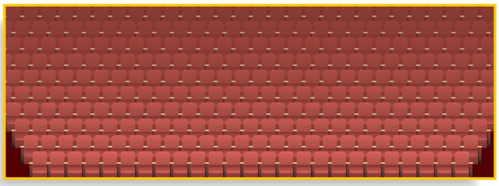
GOAL 2 ARITHMETIC SEQUENCES AND SERIES IN REAL LIFE

EXAMPLE 6 *Writing an Arithmetic Sequence*

SEATING CAPACITY The first row of a concert hall has 25 seats, and each row after the first has one more seat than the row before it. There are 32 rows of seats.

a. Write a rule for the number of seats in the nth row.

b. Thirty-five students from a class want to sit in the same row. How close to the front can they sit?



SOLUTION

a. Use $a_1 = 25$ and $d = 1$ to write a rule for a_n.

$$a_n = a_1 + (n - 1)d = 25 + (n - 1)(1) = 24 + n$$

b. Using the rule $a_n = 24 + n$, let $a_n = 35$ and solve for n.

$35 = 24 + n$ Substitute for a_n.

$11 = n$ Solve for n.

▶ The class can sit in the 11th row.

REAL LIFE *Seating Capacity*

EXAMPLE 7 *Finding the Sum of an Arithmetic Series*

Use the information about the concert hall in Example 6.

a. What is the total number of seats in the concert hall?

b. Suppose 12 more rows of seats are built (where each row has one more seat than the row before it). How many additional seats will the concert hall have?

SOLUTION

a. Find the sum of an arithmetic series with $a_1 = 25$ and $a_{32} = 24 + 32 = 56$.

$$S_{32} = 32\left(\frac{a_1 + a_{32}}{2}\right)$$ Write rule for S_{32}.

$$= 32\left(\frac{25 + 56}{2}\right) = 1296$$ Substitute for a_1 and a_{32}.

▶ There are 1296 seats in the concert hall.

b. The expanded concert hall has $32 + 12 = 44$ rows of seats. Because $a_{44} = 24 + 44 = 68$, the *total* number of seats in the expanded hall is:

$$S_{44} = 44\left(\frac{a_1 + a_{44}}{2}\right)$$ Write rule for S_{44}.

$$= 44\left(\frac{25 + 68}{2}\right) = 2046$$ Substitute for a_1 and a_{44}.

▶ The number of *additional* seats is $S_{44} - S_{32} = 2046 - 1296 = 750$.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: The expression formed by adding the terms of an arithmetic sequence is called a(n) _?_.

Concept Check ✓

2. What is the difference between an arithmetic sequence and an arithmetic series?

3. Explain how to find the sum of the first n terms of an arithmetic series.

Skill Check ✓

Write a rule for the *n*th term of the arithmetic sequence.

4. $d = 2, a_1 = 5$ **5.** $d = -3, a_2 = 18$ **6.** $d = \frac{1}{2}, a_5 = 20$

7. $a_8 = 12, a_{15} = 61$ **8.** $a_5 = 10, a_{12} = 24$ **9.** $a_{10} = 8, a_{16} = 32$

Find the sum of the first 10 terms of the arithmetic series.

10. $2 + 6 + 10 + 14 + 18 + \cdots$ **11.** $3 + \frac{7}{2} + 4 + \frac{9}{2} + 5 + \cdots$

12. $6 + 3 + 0 + (-3) + (-6) + \cdots$ **13.** $0.7 + 1.9 + 3.1 + 4.3 + 5.5 + \cdots$

14. **MOVIE THEATER** Suppose a movie theater has 42 rows of seats and there are 29 seats in the first row. Each row after the first has two more seats than the row before it. How many seats are in the theater?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 955.

IDENTIFYING ARITHMETIC SEQUENCES **Decide whether the sequence is arithmetic. Explain why or why not.**

15. 14, 11, 8, 5, 2, . . . **16.** 1, 3, 9, 27, 81, . . . **17.** $-5, -7, -11, -13, -15, \ldots$

18. 0.5, 1, 1.5, 2, 2.5, . . . **19.** $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{8}{5}, \frac{16}{5}, \ldots$ **20.** $-\frac{5}{3}, -1, -\frac{1}{3}, \frac{1}{3}, 1, \ldots$

WRITING TERMS **Write a rule for the *n*th term of the arithmetic sequence. Then find a_{25}.**

21. 1, 3, 5, 7, 9, . . . **22.** 6, 14, 22, 30, 38, . . . **23.** 9, 23, 37, 51, 65, . . .

24. $-1, 0, 1, 2, 3, \ldots$ **25.** $4, 1, -2, -5, -8, \ldots$ **26.** $\frac{1}{2}, 3, \frac{11}{2}, 8, \frac{21}{2}, \ldots$

27. $\frac{11}{2}, \frac{25}{6}, \frac{17}{6}, \frac{3}{2}, \frac{1}{6}, \ldots$ **28.** $\frac{5}{2}, \frac{11}{6}, \frac{7}{6}, \frac{1}{2}, -\frac{1}{6}, \ldots$ **29.** 1.6, 4, 6.4, 8.8, 11.2, . . .

WRITING RULES **Write a rule for the *n*th term of the arithmetic sequence.**

30. $d = 4, a_{14} = 46$ **31.** $d = -12, a_1 = 80$ **32.** $d = \frac{5}{3}, a_8 = 24$

33. $a_5 = 17, a_{15} = 77$ **34.** $d = -6, a_{12} = -4$ **35.** $a_2 = -28, a_{20} = 52$

36. $a_1 = -2, a_9 = -\frac{1}{6}$ **37.** $a_7 = 34, a_{18} = 122$ **38.** $d = -4.1, a_{16} = 48.2$

GRAPHING SEQUENCES **Graph the arithmetic sequence.**

39. $a_n = 7 + 2n$ **40.** $a_n = -3 + 5n$ **41.** $a_n = 5 - 2n$

42. $a_n = 2 - \frac{1}{3}n$ **43.** $a_n = 4 - \frac{1}{2}n$ **44.** $a_n = -0.25 + 0.45n$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 15–20
Example 2: Exs. 21–29
Example 3: Exs. 30–32, 34, 38–44
Example 4: Exs. 33, 35–37
Example 5: Exs. 45–56
Examples 6, 7: Exs. 57–60

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 45–50.

FINDING SUMS For part (a), find the sum of the first n terms of the arithmetic series. For part (b), find n for the given sum S_n.

45. $3 + 8 + 13 + 18 + 23 + \cdots$
  a. $n = 20$ **b.** $S_n = 366$

46. $50 + 42 + 34 + 26 + 18 + \cdots$
  a. $n = 40$ **b.** $S_n = 182$

47. $-10 + (-5) + 0 + 5 + 10 + \cdots$
  a. $n = 19$ **b.** $S_n = 375$

48. $34 + 31 + 28 + 25 + 22 + \cdots$
  a. $n = 32$ **b.** $S_n = -12$

49. $2 + 9 + 16 + 23 + 30 + \cdots$
  a. $n = 68$ **b.** $S_n = 1661$

50. $2 + 16 + 30 + 44 + 58 + \cdots$
  a. $n = 24$ **b.** $S_n = 2178$

USING SUMMATION NOTATION Find the sum of the series.

51. $\sum_{i=1}^{20} (3 + 5i)$ **52.** $\sum_{i=1}^{34} (1 + 8i)$ **53.** $\sum_{i=1}^{15} (-10 - 3i)$

54. $\sum_{i=1}^{22} \left(6 - \frac{3}{4}i\right)$ **55.** $\sum_{i=1}^{45} (11 + 4i)$ **56.** $\sum_{i=1}^{18} (8.1 + 4.4i)$

FOCUS ON CAREERS

ENTOMOLOGIST
Entomology is the scientific study of insects. Some entomologists study bee population dynamics and pollination of plants.

CAREER LINK
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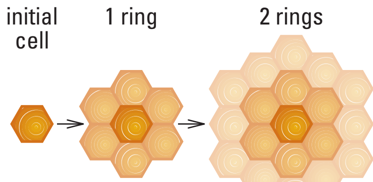
57. HONEYCOMBS Domestic bees make their honeycomb by starting with a single hexagonal cell, then forming ring after ring of hexagonal cells around the initial cell, as shown. The numbers of cells in successive rings form an arithmetic sequence.

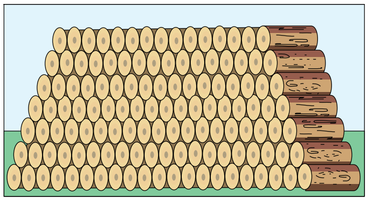


▶ Source: USDA's Carl Hayden Bee Research Lab

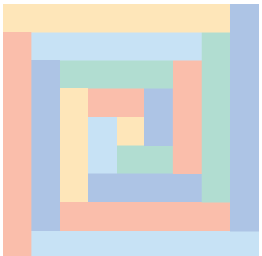
a. Write a rule for the number of cells in the nth ring.

b. What is the total number of cells in the honeycomb after the 9th ring is formed? (*Hint:* Do not forget to count the initial cell.)

58. STACKING LOGS Logs are stacked in a pile, as shown at the right. The bottom row has 21 logs and the top row has 15 logs. Each row has one less log than the row below it. How many logs are in the pile?



59. QUILTING A quilt is made up of strips of cloth, starting with an inner square surrounded by rectangles to form successively larger squares. The inner square and all rectangles have a width of 1 foot. Write an expression using summation notation that gives the sum of the areas of all the strips of cloth used to make the quilt shown. Then evaluate the expression.



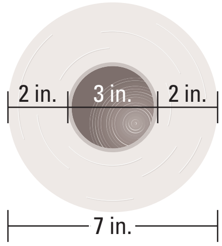
60. SEATING REVENUE Suppose each seat in rows 1 through 11 of the concert hall in Example 6 costs \$24, each seat in rows 12 through 22 costs \$18, and each seat in rows 23 through 32 costs \$12. How much money does the concert hall take in for a sold-out event?

61. *Writing* Compare the graphs of $a_n = 2n + 1$ where n is a positive integer and $f(x) = 2x + 1$ where x is a real number. Discuss how the graph of an arithmetic sequence is similar to and different from the graph of a linear function.

Test Preparation

62. MULTI-STEP PROBLEM A paper manufacturer sells paper rolled onto cardboard dowels. The thickness of the paper is 0.004 inch. The diameter of a dowel is 3 inches, and the total diameter of a roll is 7 inches.

n	d_n (in.)	l_n (in.)
1	3	3π
2	?	?
3	?	?
4	?	?



a. Let n be the number of times the paper is wrapped around the dowel, let d_n be the diameter of the roll just before the nth wrap, and let l_n be the length of paper added in the nth wrap. Copy and complete the table.

b. What can you say about the sequence $l_1, l_2, l_3, l_4, \ldots$? Write a formula for the nth term of the sequence.

c. Find the number of times the paper must be wrapped around the dowel to create a roll with a 7 inch diameter. Use your answer and the formula from part (b) to find the length of paper in a roll with a 7 inch diameter.

d. LOGICAL REASONING Suppose a roll with a 7 inch diameter costs $15. How much would you expect to pay for a roll with an 11 inch diameter whose dowel also has a diameter of 3 inches? Explain your reasoning and any assumptions you make.

★ Challenge

63. AHMES PAPYRUS One of the major sources of our knowledge of Egyptian mathematics is the Ahmes papyrus (also known as the Rhind papyrus), which is a scroll copied in 1650 B.C. by an Egyptian scribe. The following problem is from the Ahmes papyrus.

Divide 10 hekats of barley among 10 men so that the common difference is $\frac{1}{8}$ of a hekat of barley.

Use what you know about arithmetic sequences and series to solve the problem.

EXTRA CHALLENGE
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MIXED REVIEW

SOLVING RATIONAL EXPONENT EQUATIONS **Solve the equation.** **(Review 7.6 for 11.3)**

64. $x^{1/2} = 5$ **65.** $2x^{3/4} = 54$ **66.** $x^{2/3} + 10 = 19$

67. $(8x)^{1/2} + 6 = 0$ **68.** $x^{1/3} - 11 = 0$ **69.** $(2x)^{1/2} = x - 4$

SOLVING EXPONENTIAL EQUATIONS **Solve the exponential equation.** **(Review 8.6 for 11.3)**

70. $2^x = 4.5$ **71.** $4^x - 3 = 5$ **72.** $10^{3x} + 7 = 15$

73. $6^x - 5 = 1$ **74.** $25^x - 28 = 97$ **75.** $5(2)^{2x} - 4 = 13$

GRAPHING EQUATIONS **Graph the equation.** **(Review 10.3)**

76. $x^2 + y^2 = 9$ **77.** $x^2 + y^2 = 24$ **78.** $x^2 + y^2 = 64$

79. $6x^2 + 6y^2 = 150$ **80.** $\frac{1}{2}x^2 + \frac{1}{2}y^2 = 4$ **81.** $20x^2 + 20y^2 = 400$