11.1 An Introduction to Sequences and Series

What *you should learn*

GOAL 1 Use and write sequences.

GOAL 2 Use summation notation to write series and find sums of series, as applied in **Example 6**.

Why *you should learn it*

▼ To model **real-life** situations, such as building a roof frame in **Exs. 65 and 66**.

REAL LIFE

GOAL 1 Using and Writing Sequences

Saying that a collection of objects is listed "in sequence" means that the collection is ordered so that it has a first member, a second member, a third member, and so on. Below are two examples of sequences of numbers. The numbers in the sequences are called **terms**.

SEQUENCE 1: 3, 6, 9, 12, 15

SEQUENCE 2: 3, 6, 9, 12, 15, . . .

You can think of a **sequence** as a function whose domain is a set of consecutive integers. If a domain is not specified, it is understood that the domain starts with 1.

DOMAIN:	1	2	3	4	5	**The domain gives the relative position of each term: 1st, 2nd, 3rd, and so on.**
RANGE:	3	6	9	12	15	**The range gives the terms of the sequence.**

Sequence 1 above is a **finite sequence** because it has a last term. Sequence 2 is an **infinite sequence** because it continues without stopping. Both sequences have the general rule $a_n = 3n$ where a_n represents the nth term of the sequence. The general rule can also be written using function notation: $f(n) = 3n$.

EXAMPLE 1 *Writing Terms of Sequences*

Write the first six terms of the sequence.

a. $a_n = 2n + 3$ **b.** $f(n) = (-2)^{n-1}$

Solution

a. $a_1 = 2(1) + 3 = 5$ 1st term

$a_2 = 2(2) + 3 = 7$ 2nd term

$a_3 = 2(3) + 3 = 9$ 3rd term

$a_4 = 2(4) + 3 = 11$ 4th term

$a_5 = 2(5) + 3 = 13$ 5th term

$a_6 = 2(6) + 3 = 15$ 6th term

b. $f(1) = (-2)^{1-1} = 1$ 1st term

$f(2) = (-2)^{2-1} = -2$ 2nd term

$f(3) = (-2)^{3-1} = 4$ 3rd term

$f(4) = (-2)^{4-1} = -8$ 4th term

$f(5) = (-2)^{5-1} = 16$ 5th term

$f(6) = (-2)^{6-1} = -32$ 6th term

Student Help

Look Back
For help with evaluating expressions, see p. 12.

If the terms of a sequence have a recognizable pattern, then you may be able to write a rule for the nth term of the sequence.

EXAMPLE 2 *Writing Rules for Sequences*

> **STUDENT HELP**
> **Study Tip**
> If you are given only the first several terms of a sequence, there is no *single* rule for the nth term. For instance, the sequence 2, 4, 8, . . . can be given by $a_n = 2^n$ or $a_n = n^2 - n + 2$.

For each sequence, describe the pattern, write the next term, and write a rule for the nth term.

a. $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$ **b.** 2, 6, 12, 20, . . .

SOLUTION

a. You can write the terms as $\left(-\frac{1}{3}\right)^1, \left(-\frac{1}{3}\right)^2, \left(-\frac{1}{3}\right)^3, \left(-\frac{1}{3}\right)^4, \ldots.$

The next term is $a_5 = \left(-\frac{1}{3}\right)^5 = -\frac{1}{243}$. A rule for the nth term is $a_n = \left(-\frac{1}{3}\right)^n$.

b. You can write the terms as 1(2), 2(3), 3(4), 4(5), ...

The next term is $f(5) = 5(6) = 30$. A rule for the nth term is $f(n) = n(n + 1)$.

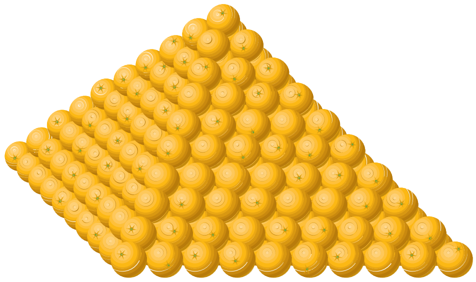
. . . .

You can graph a sequence by letting the horizontal axis represent the position numbers (the domain) and the vertical axis represent the terms (the range).

REAL LIFE
Retail Displays

EXAMPLE 3 *Graphing a Sequence*

You work in the produce department of a grocery store and are stacking oranges in the shape of a square pyramid with 10 layers.



a. Write a rule for the number of oranges in each layer.

b. Graph the sequence.

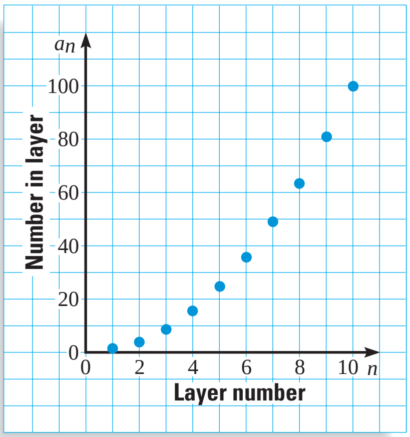
SOLUTION

a. The diagram below shows the first three layers of the stack. Let a_n represent the number of oranges in layer n.

n	1	2	3
a_n	$1 = 1^2$	$4 = 2^2$	$9 = 3^2$

From the diagram, you can see that $a_n = n^2$.

b. Plot the points (1, 1), (2, 4), (3, 9), . . . , (10, 100). The graph is shown at the right.



> **STUDENT HELP**
> **Study Tip**
> Although the plotted points in part (b) of Example 3 follow a curve, do *not* draw the curve because the sequence is defined only for integer values of n.

GOAL 2 USING SERIES

When the terms of a sequence are added, the resulting expression is a **series**. A series can be infinite or finite.

FINITE SEQUENCE	INFINITE SEQUENCE
3, 6, 9, 12, 15	3, 6, 9, 12, 15, . . .
FINITE SERIES	**INFINITE SERIES**
$3 + 6 + 9 + 12 + 15$	$3 + 6 + 9 + 12 + 15 + \cdots$

You can use **summation notation** to write a series. For example, for the finite series shown above, you can write

$$3 + 6 + 9 + 12 + 15 = \sum_{i=1}^{5} 3i$$

where i is the *index of summation*, 1 is the *lower limit of summation*, and 5 is the *upper limit of summation*. In this case the summation notation is read as "the sum from i equals 1 to 5 of $3i$." Summation notation is also called **sigma notation** because it uses the uppercase Greek letter *sigma,* written Σ.

Summation notation for an infinite series is similar to that for a finite series. For example, for the infinite series shown above, you can write:

$$3 + 6 + 9 + 12 + 15 + \cdots = \sum_{i=1}^{\infty} 3i$$

The infinity symbol, ∞, indicates that the series continues without end.

EXAMPLE 4 *Writing Series with Summation Notation*

Write each series with summation notation.

a. $5 + 10 + 15 + \cdots + 100$

b. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

SOLUTION

a. Notice that the first term is 5(1), the second is 5(2), the third is 5(3), and the last is 5(20). So, the terms of the series can be written as:

$$a_i = 5i \text{ where } i = 1, 2, 3, \ldots, 20$$

▶ The summation notation for the series is $\sum_{i=1}^{20} 5i$.

b. Notice that for each term the denominator of the fraction is 1 more than the numerator. So, the terms of the series can be written as:

$$a_i = \frac{i}{i+1} \text{ where } i = 1, 2, 3, 4, \ldots$$

▶ The summation notation for the series is $\sum_{i=1}^{\infty} \frac{i}{i+1}$.

The index of summation does not have to be i — any letter can be used. Also, the index does not have to begin at 1. For instance, in part (b) of Example 5 on the next page, the index begins at 3.

EXAMPLE 5 *Using Summation Notation*

Find the sum of the series.

a. $\sum_{i=1}^{6} 2i = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6)$

$= 2(1 + 2 + 3 + 4 + 5 + 6)$

$= 2(21)$

$= 42$

b. $\sum_{k=3}^{6} (2 + k^2) = (2 + 3^2) + (2 + 4^2) + (2 + 5^2) + (2 + 6^2)$

$= 11 + 18 + 27 + 38$

$= 94$

STUDENT HELP

Study Tip
Notice that the first term in Example 5b occurs when $k = 3$ (not when $k = 1$) and that there are only 4 terms (not 6) in the series.

The sum of the terms of a finite sequence can be found by simply adding the terms. For sequences with many terms, however, adding the terms can be tedious. Formulas for finding the sum of the terms of three special types of sequences are given below.

CONCEPT SUMMARY **FORMULAS FOR SPECIAL SERIES**

1. $\sum_{i=1}^{n} 1 = n$ **2.** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ **3.** $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

In words, the first formula gives the sum of n 1's. The second formula gives the sum of the positive integers from 1 to n. The third formula gives the sum of the squares of the positive integers from 1 to n.

EXAMPLE 6 *Using a Formula for a Sum*

RETAIL DISPLAYS How many oranges are in the stack in Example 3?

SOLUTION

From Example 3 you know that the ith term of the series is given by $a_i = i^2$, where $i = 1, 2, 3, \ldots, 10$. Using summation notation and the third formula listed above, you can find the total number of oranges as follows.

$$\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + \cdots + 10^2$$

$$= \frac{10(10+1)(2 \cdot 10 + 1)}{6}$$

$$= \frac{10(11)(21)}{6}$$

$$= 385$$

▶ There are 385 oranges in the stack. Check this by actually adding the number of oranges in each of the ten layers.

FOCUS ON PEOPLE

THOMAS HALES, a mathematician at the University of Michigan, proved in 1998 that the arrangement of identical spheres illustrated in Examples 3 and 6 (using oranges) wastes less space than any other arrangement.

Guided Practice

Vocabulary Check ✓

1. Explain the difference between a sequence and a series.

Concept Check ✓

2. Answer the following questions about the series $\sum_{k=3}^{10} (k+2)$.
   a. In words, how do you read the summation notation?
   b. What is the index of summation?
   c. What is the lower limit of summation?
   d. What is the upper limit of summation?

Skill Check ✓

Write the first six terms of the sequence.

3. $a_n = 2n$
4. $a_n = 6 - n$
5. $a_n = 3n + 1$
6. $f(n) = 2^{n+3}$
7. Find the sum of the series in Exercise 2.
8. **STACKING** Find the total number of oranges in the stack in Example 3 if there are 12 layers.

Practice and Applications

STUDENT HELP

Extra Practice to help you master skills is on p. 955.

WRITING TERMS **Write the first six terms of the sequence.**

9. $a_n = n + 1$
10. $a_n = n^2$
11. $a_n = 3 - n$
12. $a_n = n^3 - 1$
13. $a_n = (n + 1)^2$
14. $a_n = (-n)^3$
15. $a_n = n^2 + 3$
16. $a_n = (n - 1)^2$
17. $f(n) = \frac{n}{n+1}$
18. $f(n) = \frac{n^2}{2n}$
19. $f(n) = \frac{n+2}{2n}$
20. $f(n) = \frac{3}{-n}$

WRITING RULES **Write the next term in the sequence. Then write a rule for the *n*th term.**

21. $1, 3, 5, 7, \ldots$
22. $1, 10, 100, 1000, \ldots$
23. $2, -4, 8, -10, 14, \ldots$
24. $-5, 10, -15, 20, \ldots$
25. $-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, \ldots$
26. $\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \ldots$
27. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ldots$
28. $\frac{1}{20}, \frac{2}{30}, \frac{3}{40}, \frac{4}{50}, \ldots$
29. $1.9, 2.7, 3.5, 4.3, 5.1, \ldots$

GRAPHING SEQUENCES **Graph the sequence.**

30. $1, 4, 7, 10, \ldots, 28$
31. $3, 6, 12, 21, 33, 48$
32. $-1, -6, -11, -16, -21$
33. $1, 4, 9, 16, 25, 36$
34. $\frac{1}{9}, \frac{2}{8}, \frac{3}{7}, \frac{4}{6}, \ldots, \frac{9}{1}$
35. $3, -6, 9, -12, \ldots, -36$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 9–20
Example 2: Exs. 21–29
Example 3: Exs. 30–35, 64–68
Example 4: Exs. 36–43
Example 5: Exs. 44–55
Example 6: Exs. 56–68

WRITING SUMMATION NOTATION **Write the series with summation notation.**

36. $1 + 5 + 9 + 13 + 17$
37. $4 + 8 + 12 + 16 + 20$
38. $-3 + 3 + 9 + 15 + 21 + \cdots$
39. $1 - 2 + 3 - 4 + 5 - \cdots$
40. $-7 - 8 - 9 - 10 - 11$
41. $\frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \frac{8}{9} + \cdots$
42. $1 + 0.1 + 0.01 + 0.001$
43. $1 + 4 + 9 + 16 + 25 + 36$

USING SUMMATION NOTATION **Find the sum of the series.**

44. $\sum_{i=1}^{6} 3i$ **45.** $\sum_{i=0}^{5} 12i$ **46.** $\sum_{n=0}^{4} n^2$ **47.** $\sum_{n=1}^{3} 4n^3$

48. $\sum_{k=1}^{5} (k^2 - 1)$ **49.** $\sum_{n=0}^{4} (2n^2 + 1)$ **50.** $\sum_{k=1}^{4} k(k + 2)$ **51.** $\sum_{n=2}^{10} \frac{2}{n}$

52. $\sum_{n=2}^{12} \frac{1}{n-1}$ **53.** $\sum_{n=1}^{5} \frac{n}{n+1}$ **54.** $\sum_{i=2}^{6} \frac{i}{i-1}$ **55.** $\sum_{n=1}^{\infty} \left(\frac{n}{n^2} - \frac{1}{n}\right)$

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 56–63.

USING FORMULAS **Use one of the formulas for special series to find the sum of the series.**

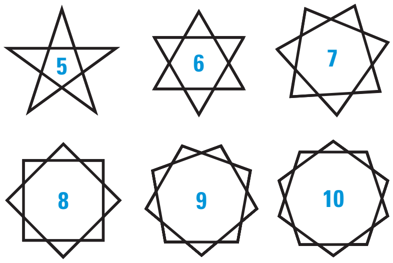
56. $\sum_{i=1}^{42} 1$ **57.** $\sum_{n=1}^{5} n$ **58.** $\sum_{i=1}^{18} i$ **59.** $\sum_{k=1}^{20} k$

60. $\sum_{n=1}^{6} n^2$ **61.** $\sum_{i=1}^{10} i^2$ **62.** $\sum_{i=1}^{12} i^2$ **63.** $\sum_{k=1}^{35} k^2$

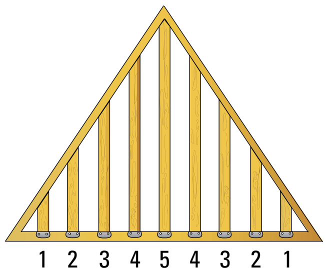
64. **GEOMETRY CONNECTION** The degree measurement d_n in each angle at the tips of the six n-pointed stars shown at the right is given by:

$$d_n = \frac{180(n-4)}{n},\ n \geq 5$$

Write the first six terms of the sequence.



CARPENTRY **In Exercises 65 and 66, use the following information.** The diagram shows part of a roof frame. The length (in feet) of each vertical support is given below the support. These lengths form an arithmetic sequence from each end to the middle.



65. Find the total length of the vertical supports from one end to the middle.

66. Use your result from Exercise 65 to find the total length of the vertical supports from end to end.

FOCUS ON APPLICATIONS

TOWER OF HANOI The puzzle was first described in print by the French mathematician Edouard Lucas in 1883 in his four volume book on recreational mathematics.

67. **TOWER OF HANOI** In the puzzle called the Tower of Hanoi, the object is to use a series of moves to take the rings from one peg and stack them in order on another peg. A move consists of moving exactly one ring, and no ring may be placed on top of a smaller ring. The minimum number of moves required to move n rings is 1 for 1 ring, 3 for 2 rings, 7 for 3 rings, 15 for 4 rings, and 31 for 5 rings. Find a formula for the sequence. What is the minimum number of moves required to move 6 rings?

68. **PYRAMID STACK** Suppose you are stacking tennis balls in a pyramid as a display at a sports store. If the base is an equilateral triangle, then the number a_n of balls per layer would be $a_n = \frac{1}{2}n^2 + \frac{1}{2}n$ where $n = 1$ represents the top layer. How many balls are in the fifth layer? How many balls are in a stack with 5 layers?

Test Preparation

QUANTITATIVE COMPARISON **In Exercises 69 and 70, choose the statement that is true about the given quantities.**

- (A) The quantity in column A is greater.
- (B) The quantity in column B is greater.
- (C) The two quantities are equal.
- (D) The relationship cannot be determined from the given information.

	Column A	Column B
69.	The fifth term of the sequence $a_n = n^2 + 1$	$\sum_{n=1}^{5} (n^2 + 1)$
70.	The first term of the sequence $a_n = 5 - n$	$\sum_{n=4}^{8} (5 - n)$

★ Challenge

71. LOGICAL REASONING Tell whether the statement about summation notation is *true* or *false*. If the statement is true, prove it. If the statement is false, give a counterexample.

a. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

b. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

c. $\sum_{i=1}^{n} a_i b_i = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$

d. $\sum_{i=1}^{n} (a_i)^k = \left(\sum_{i=1}^{n} a_i\right)^k$

EXTRA CHALLENGE
www.mcdougallittell.com

72. Using the true statements from Exercise 71 and the special formulas from page 654, find a formula for the number of balls in n layers of the pyramid in Exercise 68.

MIXED REVIEW

SOLVING EQUATIONS **Solve the equation. Check your solution.** (Review 1.3 for 11.2)

73. $17 = 3x + 5$ **74.** $18 = -7 + x$ **75.** $15 = -1 + 8x$

76. $9 = 4 - 5x$ **77.** $5 = 6 - 2x$ **78.** $24 = 10 + 7x$

FINDING EXPONENTIAL MODELS **Use the table of values to draw a scatter plot of ln *y* versus *x*. Then find an exponential model for the data.** (Review 8.7)

79.

x	1	2	3	4	5	6	7	8	9
y	5	10	20	40	80	160	320	640	1280

80.

x	1	2	3	4	5	6	7	8
y	3.2	9.6	28.8	86.4	259.2	777.6	2332.8	6998.4

FINDING THE DISTANCE **Find the distance between the points.** (Review 10.1)

81. $(0, 0), (-4, -6)$ **82.** $(1, 4), (-3, -9)$ **83.** $(5, 2), (-1, 8)$

84. $(9, -1), (2, 9)$ **85.** $(3, -3), (11, -4)$ **86.** $(10, 30), (40, -20)$