10.5 Hyperbolas

What *you should learn*

GOAL 1 Graph and write equations of hyperbolas.

GOAL 2 Use hyperbolas to solve **real-life** problems, such as modeling a sundial in **Exs. 64–66**.

Why *you should learn it*

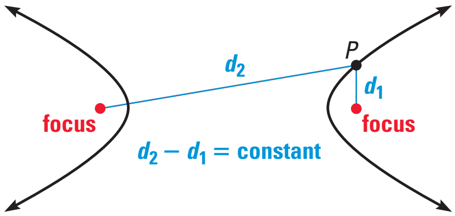
▼ To model **real-life** objects, such as a sculpture in **Example 4**.

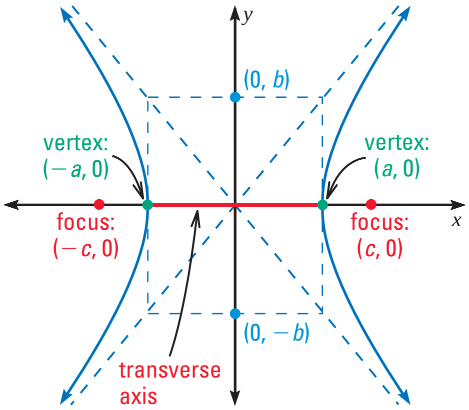
REAL LIFE

GOAL 1 GRAPHING AND WRITING EQUATIONS OF HYPERBOLAS

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, recall that the *sum* of the distances between a point on the ellipse and the two foci is constant. For a hyperbola, the *difference* is constant.

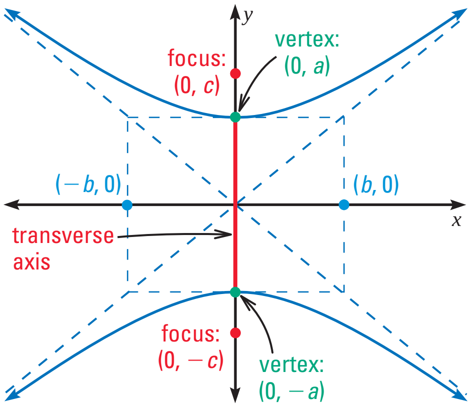
A **hyperbola** is the set of all points P such that the difference of the distances from P to two fixed points, called the **foci**, is constant. The line through the foci intersects the hyperbola at two points, the **vertices**. The line segment joining the vertices is the **transverse axis**, and its midpoint is the **center** of the hyperbola. A hyperbola has two branches and two asymptotes. The asymptotes contain the diagonals of a rectangle centered at the hyperbola's center, as shown below.





Hyperbola with horizontal transverse axis

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Hyperbola with vertical transverse axis

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

CHARACTERISTICS OF A HYPERBOLA (CENTER AT ORIGIN)

The **standard form of the equation of a hyperbola** with center at (0, 0) is as follows.

EQUATION	TRANSVERSE AXIS	ASYMPTOTES	VERTICES
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Horizontal	$y = \pm\frac{b}{a}x$	$(\pm a, 0)$
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	Vertical	$y = \pm\frac{a}{b}x$	$(0, \pm a)$

The foci of the hyperbola lie on the transverse axis, c units from the center where $c^2 = a^2 + b^2$.

EXAMPLE 1 *Graphing an Equation of a Hyperbola*

Draw the hyperbola given by $4x^2 - 9y^2 = 36$.

SOLUTION

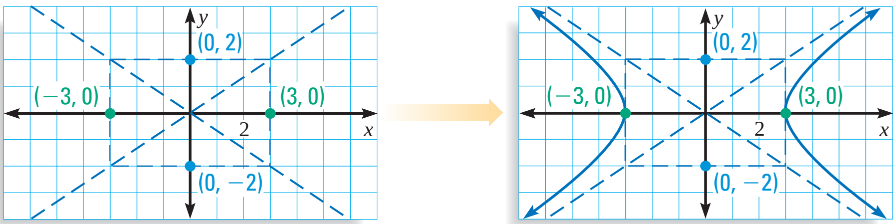
First rewrite the equation in standard form.

$4x^2 - 9y^2 = 36$ **Write original equation.**

$\frac{4x^2}{36} - \frac{9y^2}{36} = \frac{36}{36}$ **Divide each side by 36.**

$\frac{x^2}{9} - \frac{y^2}{4} = 1$ **Simplify.**

Note from the equation that $a^2 = 9$ and $b^2 = 4$, so $a = 3$ and $b = 2$. Because the x^2-term is positive, the transverse axis is horizontal and the vertices are at **(−3, 0)** and **(3, 0)**. To draw the hyperbola, first draw a rectangle that is centered at the origin, $2a = 6$ units wide and $2b = 4$ units high. Then show the asymptotes by drawing the lines that pass through opposite corners of the rectangle. Finally, draw the hyperbola.

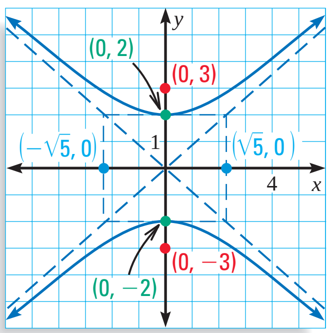


EXAMPLE 2 *Writing an Equation of a Hyperbola*

Write an equation of the hyperbola with foci at (0, −3) and (0, 3) and vertices at (0, −2) and (0, 2).

SOLUTION

The transverse axis is vertical because the foci and vertices lie on the y-axis. The center is the origin because the foci and the vertices are equidistant from the origin. Since the foci are each 3 units from the center, $c = 3$. Similarly, because the vertices are each 2 units from the center, $a = 2$.



You can use these values of a and c to find b.

$b^2 = c^2 - a^2$

$b^2 = 3^2 - 2^2 = 9 - 4 = 5$

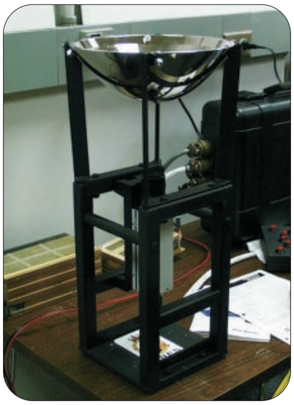
$b = \sqrt{5}$

Because the transverse axis is vertical, the standard form of the equation is as follows.

$\frac{y^2}{2^2} - \frac{x^2}{(\sqrt{5})^2} = 1$ **Substitute 2 for *a* and $\sqrt{5}$ for *b*.**

$\frac{y^2}{4} - \frac{x^2}{5} = 1$ **Simplify.**

FOCUS ON APPLICATIONS



PANORAMIC CAMERAS
A panoramic photograph taken with the camera shown above gives a 360° view of a scene.

APPLICATION LINK
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GOAL 2 USING HYPERBOLAS IN REAL LIFE

EXAMPLE 3 *Using a Real-Life Hyperbola*

PHOTOGRAPHY A hyperbolic mirror can be used to take panoramic photographs. A camera is pointed toward the vertex of the mirror and is positioned so that the lens is at one focus of the mirror. An equation for the cross section of the mirror is $\frac{y^2}{16} - \frac{x^2}{9} = 1$ where x and y are measured in inches. How far from the mirror is the lens?

SOLUTION

Notice from the equation that $a^2 = 16$ and $b^2 = 9$, so $a = 4$ and $b = 3$. Use these values and the equation $c^2 = a^2 + b^2$ to find the value of c.

$c^2 = a^2 + b^2$ — **Equation relating *a*, *b*, and *c***

$c^2 = 16 + 9 = 25$ — **Substitute for *a* and *b* and simplify.**

$c = 5$ — **Solve for *c*.**

Since $a = 4$ and $c = 5$, the vertices are at $(0, -4)$ and $(0, 4)$ and the foci are at $(0, -5)$ and $(0, 5)$. The camera is below the mirror, so the lens is at $(0, -5)$ and the vertex of the mirror is at $(0, 4)$. The distance between these points is $4 - (-5) = 9$.

▶ The lens is 9 inches from the mirror.

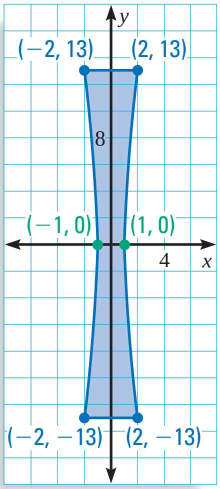
REAL LIFE Sculpture

EXAMPLE 4 *Modeling with a Hyperbola*

The diagram at the right shows the hyperbolic cross section of a sculpture located at the Fermi National Accelerator Laboratory in Batavia, Illinois.

a. Write an equation that models the curved sides of the sculpture.

b. At a height of 5 feet, how wide is the sculpture? (Each unit in the coordinate plane represents 1 foot.)



SOLUTION

a. From the diagram you can see that the transverse axis is horizontal and $a = 1$. So the equation has this form:

$$\frac{x^2}{1^2} - \frac{y^2}{b^2} = 1$$

Because the hyperbola passes through the point $(2, 13)$, you can substitute $x = 2$ and $y = 13$ into the equation and solve for b. When you do this, you obtain $b \approx 7.5$.

▶ An equation of the hyperbola is $\frac{x^2}{1^2} - \frac{y^2}{(7.5)^2} = 1$.

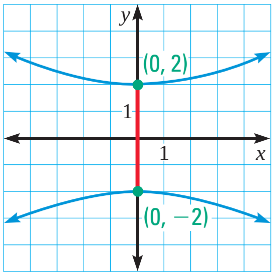
b. At a height of 5 feet above the ground, $y = -8$. To find the width of the sculpture, substitute this value into the equation and solve for x. You get $x \approx 1.46$.

▶ At a height of 5 feet, the width is $2x \approx 2.92$ feet.

Guided Practice

Vocabulary Check ✓

1. Complete these statements: The points $(0, -2)$ and $(0, 2)$ in the graph at the right are the _?_ of the hyperbola. The segment joining these two points is the _?_.



Ex. 1

Concept Check ✓

2. How are the definitions of ellipse and hyperbola alike? How are they different?

3. How do the asymptotes of a hyperbola help you draw the hyperbola?

Skill Check ✓

Graph the equation. Identify the foci and asymptotes.

4. $\frac{x^2}{49} - \frac{y^2}{81} = 1$

5. $\frac{y^2}{100} - \frac{x^2}{75} = 1$

6. $\frac{x^2}{64} - y^2 = 1$

7. $36x^2 - 4y^2 = 144$

8. $12y^2 - 25x^2 = 300$

9. $y^2 - 9x^2 = 9$

Write an equation of the hyperbola with the given foci and vertices.

10. Foci: $(0, -5), (0, 5)$
Vertices: $(0, -3), (0, 3)$

11. Foci: $(-8, 0), (8, 0)$
Vertices: $(-7, 0), (7, 0)$

12. Foci: $(-\sqrt{34}, 0), (\sqrt{34}, 0)$
Vertices: $(-5, 0), (5, 0)$

13. Foci: $(0, -9), (0, 9)$
Vertices: $(0, -3\sqrt{5}), (0, 3\sqrt{5})$

14. **PHOTOGRAPHY** Look back at Example 3. Suppose a mirror has a cross section modeled by the equation $\frac{x^2}{25} - \frac{y^2}{9} = 1$ where x and y are measured in inches. If you place a camera with its lens at the focus, how far is the lens from the vertex of the mirror?

Practice and Applications

STUDENT HELP

Extra Practice
to help you master skills is on p. 954.

MATCHING **Match the equation with its graph.**

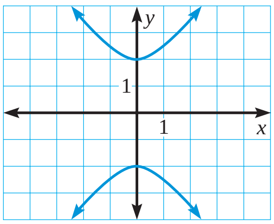
15. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

16. $\frac{y^2}{4} - \frac{x^2}{2} = 1$

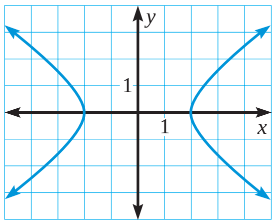
17. $\frac{y^2}{16} - \frac{x^2}{4} = 1$

18. $\frac{x^2}{4} - \frac{y^2}{2} = 1$

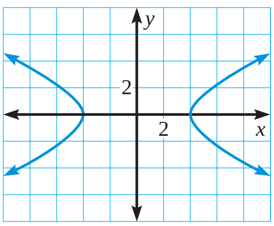
A.



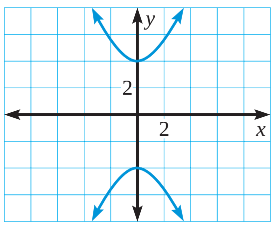
B.



C.



D.



STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 15–55
Example 2: Exs. 56–63
Example 3: Ex. 67
Example 4: Exs. 64–66

STANDARD FORM **Write the equation of the hyperbola in standard form.**

19. $36x^2 - 9y^2 = 324$ **20.** $y^2 - 81x^2 = 81$ **21.** $36y^2 - 4x^2 = 9$

22. $16y^2 - 36x^2 + 9 = 0$ **23.** $y^2 - \frac{x^2}{36} = 4$ **24.** $\frac{x^2}{9} - \frac{4y^2}{9} = 9$

IDENTIFYING PARTS **Identify the vertices and foci of the hyperbola.**

25. $\frac{x^2}{9} - \frac{y^2}{64} = 1$ **26.** $\frac{y^2}{49} - x^2 = 1$ **27.** $\frac{x^2}{121} - \frac{y^2}{4} = 1$

28. $4y^2 - 81x^2 = 324$ **29.** $25y^2 - 4x^2 = 100$ **30.** $36x^2 - 10y^2 = 360$

GRAPHING **Graph the equation. Identify the foci and asymptotes.**

31. $\frac{x^2}{25} - \frac{y^2}{121} = 1$ **32.** $\frac{x^2}{36} - y^2 = 1$ **33.** $\frac{y^2}{25} - \frac{x^2}{49} = 1$

34. $\frac{y^2}{9} - \frac{x^2}{100} = 1$ **35.** $\frac{x^2}{169} - \frac{y^2}{16} = 1$ **36.** $\frac{y^2}{64} - x^2 = 1$

37. $\frac{16x^2}{25} - \frac{y^2}{81} = 1$ **38.** $\frac{x^2}{144} - \frac{y^2}{121} = 1$ **39.** $\frac{x^2}{64} - \frac{9y^2}{4} = 1$

40. $\frac{y^2}{25} - \frac{x^2}{16} = 16$ **41.** $100x^2 - 81y^2 = 8100$ **42.** $x^2 - 9y^2 = 25$

GRAPHING HYPERBOLAS **Use a graphing calculator to graph the equation. Tell what two equations you entered into the calculator.**

43. $\frac{y^2}{144} - \frac{x^2}{100} = 1$ **44.** $\frac{x^2}{16} - \frac{y^2}{25} = 1$ **45.** $\frac{x^2}{42.25} - \frac{y^2}{72.25} = 1$

46. $\frac{y^2}{2.73} - \frac{x^2}{3.58} = 1$ **47.** $\frac{x^2}{10.1} - \frac{y^2}{22.3} = 1$ **48.** $1.2x^2 - 8.5y^2 = 4.6$

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 49.

49. CRITICAL THINKING Suppose you tried to graph an equation of a hyperbola on a graphing calculator. You enter one function correctly, but you forget to enter the other function. Sketch what your graph might look like if the transverse axis is horizontal. Then sketch what your graph might look like if the transverse axis is vertical.

GRAPHING CONIC SECTIONS **In Exercises 50–55, the equations of parabolas, circles, ellipses, and hyperbolas are given. Graph the equation.**

50. $\frac{x^2}{169} - \frac{y^2}{25} = 1$ **51.** $x^2 + y^2 = 30$ **52.** $\frac{y^2}{9} - \frac{x^2}{64} = 1$

53. $x^2 = 15y$ **54.** $\frac{x^2}{196} + \frac{y^2}{256} = 1$ **55.** $14x^2 + 14y^2 = 126$

WRITING EQUATIONS **Write an equation of the hyperbola with the given foci and vertices.**

56. Foci: $(0, -13), (0, 13)$
Vertices: $(0, -5), (0, 5)$

57. Foci: $(-8, 0), (8, 0)$
Vertices: $(-6, 0), (6, 0)$

58. Foci: $(-4, 0), (4, 0)$
Vertices: $(-1, 0), (1, 0)$

59. Foci: $(-6, 0), (6, 0)$
Vertices: $(-5, 0), (5, 0)$

60. Foci: $(0, -7), (0, 7)$
Vertices: $(0, -3), (0, 3)$

61. Foci: $(0, -9), (0, 9)$
Vertices: $(0, -8), (0, 8)$

62. Foci: $(-8, 0), (8, 0)$
Vertices: $\left(-4\sqrt{3}, 0\right), \left(4\sqrt{3}, 0\right)$

63. Foci: $\left(0, -5\sqrt{6}\right), \left(0, 5\sqrt{6}\right)$
Vertices: $(0, -4), (0, 4)$

FOCUS ON APPLICATIONS

SUNDIAL The Richard D. Swensen sundial, located at the University of Wisconsin – River Falls, gives the correct time to the minute.

APPLICATION LINK www.mcdougallittell.com

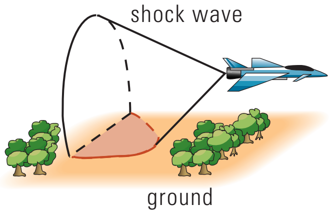
SUNDIAL In Exercises 64–66, use the following information.
The sundial at the left was designed by Professor John Shepherd. The shadow of the *gnomon* traces a hyperbola throughout the day. Aluminum rods form the hyperbolas traced on the summer solstice, June 21, and the winter solstice, December 21.

64. One focus of the summer solstice hyperbola is 207 inches above the ground. The vertex of the aluminum branch is 266 inches above the ground. If the x-axis is 355 inches above the ground and the center of the hyperbola is at the origin, write an equation for the summer solstice hyperbola.

65. One focus of the winter solstice hyperbola is 419 inches above the ground. The vertex of the aluminum branch is 387 inches above the ground and the center of the hyperbola is at the origin. If the x-axis is 355 inches above the ground, write an equation for the winter solstice hyperbola.

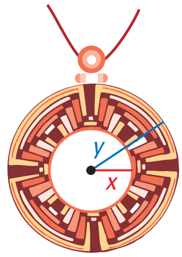
66. Use your equations from Exercises 64 and 65 to draw the lower branch of the summer solstice hyperbola and the upper branch of the winter solstice hyperbola.

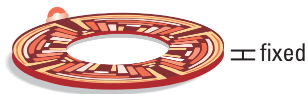
67. AERONAUTICS When an airplane travels faster than the speed of sound, the sound waves form a cone behind the airplane. If the airplane is flying parallel to the ground, the sound waves intersect the ground in a hyperbola with the airplane directly above its center. A sonic boom is heard along the hyperbola. If you hear a sonic boom that is audible along a hyperbola with the equation $\frac{x^2}{100} - \frac{y^2}{4} = 1$ where x and y are measured in miles, what is the shortest horizontal distance you could be to the airplane?



Test Preparation

68. MULTI-STEP PROBLEM Suppose you are making a ring out of clay for a necklace. If you have a fixed volume of clay and you want the ring to have a certain thickness, the area of the ring becomes fixed. However, you can still vary the inner radius x and the outer radius y.





a. Suppose you want to make a ring with an area of 2 square inches. Write an equation relating x and y.

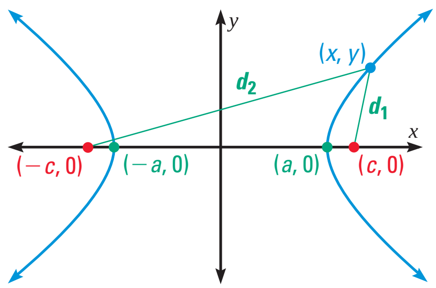
b. Find three coordinate pairs (x, y) that satisfy the relationship from part (a). Then find the width of the ring, $y - x$, for each coordinate pair.

c. *Writing* How does the width of the ring, $y - x$, change as x and y both increase? Explain why this makes sense.

★ Challenge

69. Use the diagram at the right to show that $|d_2 - d_1| = 2a$.

70. LOCATING AN EXPLOSION Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds after Microphone B. Is this enough information to decide where the sound came from? Use the fact that sound travels at 1100 feet per second.



Ex. 69

EXTRA CHALLENGE
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Mixed Review

GRAPHING FUNCTIONS **Graph the function.** (Review 2.8, 5.1 for 10.6)

71. $y = 2|x + 4| + 1$ **72.** $y = |x - 4| + 5$ **73.** $y = -|x - 6| - 8$

74. $y = 3(x - 1)^2 + 7$ **75.** $y = -2(x - 3)^2 - 6$ **76.** $y = \frac{1}{2}(x + 4)^2 + 5$

WRITING FUNCTIONS **Write a polynomial function of least degree that has real coefficients, the given zeros, and a leading coefficient of 1.** (Review 6.7)

77. 3, 1, 2 **78.** $-7, -1, 3$ **79.** $6, -2, 2$

80. $-6, 4, 2$ **81.** $5, i, -i$ **82.** $3, -3, 2i$

EVALUATING LOGARITHMIC EXPRESSIONS **Evaluate the expression without using a calculator.** (Review 8.4)

83. $\log 10{,}000$ **84.** $\log_3 27$ **85.** $\log_5 625$ **86.** $\log_2 128$

87. $\log_4 64$ **88.** $\log_3 243$ **89.** $\log_6 216$ **90.** $\log_{100} 100{,}000{,}000$

91. **TEST SCORES** Find the mean, median, mode(s), and range of the following set of test scores. (Review 7.7)

63, 67, 72, 75, 77, 78, 81, 81, 85, 86, 89, 89, 91, 92, 99

Quiz 2

Self-Test for Lessons 10.4 and 10.5

Write an equation of the ellipse with the given characteristics and center at (0, 0). (Lesson 10.4)

1. Vertex: (0, 7)
Co-vertex: $(-3, 0)$

2. Vertex: $(-6, 0)$
Co-vertex: $(0, -1)$

3. Vertex: $(-10, 0)$
Focus: (6, 0)

4. Vertex: (0, 5)
Focus: $(0, \sqrt{17})$

5. Co-vertex: $(0, 2\sqrt{3})$
Focus: $(-\sqrt{3}, 0)$

6. Co-vertex: $(-9, 0)$
Focus: (0, 4)

Graph the equation. Identify the vertices, co-vertices, and foci. (Lesson 10.4)

7. $\frac{x^2}{4} + \frac{y^2}{49} = 1$ **8.** $\frac{x^2}{6} + y^2 = 1$ **9.** $x^2 + 9y^2 = 36$

Write an equation of the hyperbola with the given characteristics. (Lesson 10.5)

10. Foci: $(0, -8), (0, 8)$
Vertices: $(0, -5), (0, 5)$

11. Foci: $(-3, 0), (3, 0)$
Vertices: $(-1, 0), (1, 0)$

12. Foci: $(-6, 0), (6, 0)$
Vertices: $(-4, 0), (4, 0)$

13. Foci: $(0, -2\sqrt{5}), (0, 2\sqrt{5})$
Vertices: $(0, -4), (0, 4)$

Graph the equation. Identify the vertices, foci, and asymptotes. (Lesson 10.5)

14. $\frac{y^2}{25} - \frac{x^2}{36} = 1$ **15.** $8y^2 - 20x^2 = 160$ **16.** $18x^2 - 4y^2 = 36$

17. **SPACE EXPLORATION** Suppose a satellite's orbit is an ellipse with Earth's center at one focus. If the satellite's least distance from Earth's surface is 150 miles and its greatest distance from Earth's surface is 600 miles, write an equation for the ellipse. (Use 4000 miles as Earth's radius.) (Lesson 10.4)