10.4 Ellipses

What you should learn

GOAL 1 Graph and write equations of ellipses.

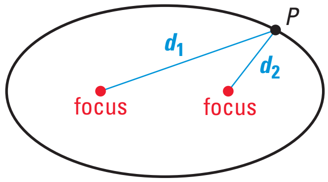
GOAL 2 Use ellipses in **real-life** situations, such as modeling the orbit of Mars in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the area of an elliptical Australian football field in **Exs. 73–75**.

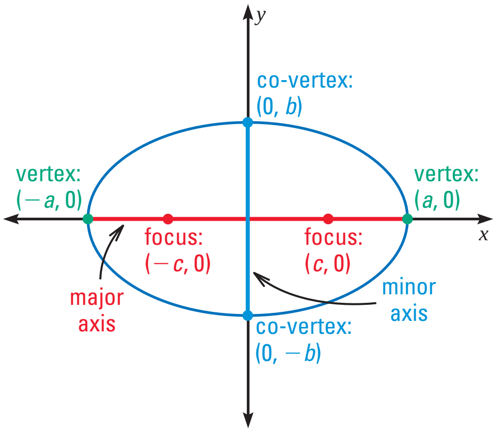
GOAL 1 GRAPHING AND WRITING EQUATIONS OF ELLIPSES

An **ellipse** is the set of all points P such that the sum of the distances between P and two distinct fixed points, called the **foci**, is a constant.



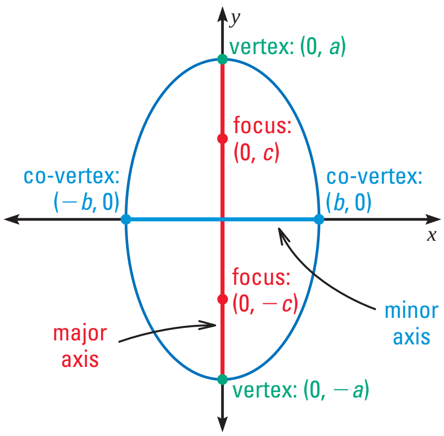
$d_1 + d_2 =$ **constant**

The line through the foci intersects the ellipse at two points, the **vertices**. The line segment joining the vertices is the **major axis**, and its midpoint is the **center** of the ellipse. The line perpendicular to the major axis at the center intersects the ellipse at two points called the **co-vertices**. The line segment that joins these points is the **minor axis** of the ellipse. The two types of ellipses we will discuss are those with a horizontal major axis and those with a vertical major axis.



Ellipse with horizontal major axis

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Ellipse with vertical major axis

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

CHARACTERISTICS OF AN ELLIPSE (CENTER AT ORIGIN)

The **standard form of the equation of an ellipse** with center at (0, 0) and major and minor axes of lengths $2a$ and $2b$, where $a > b > 0$, is as follows.

EQUATION	MAJOR AXIS	VERTICES	CO-VERTICES
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Horizontal	$(\pm a, 0)$	$(0, \pm b)$
$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$	Vertical	$(0, \pm a)$	$(\pm b, 0)$

The foci of the ellipse lie on the major axis, c units from the center where $c^2 = a^2 - b^2$.

EXAMPLE 1 *Graphing an Equation of an Ellipse*

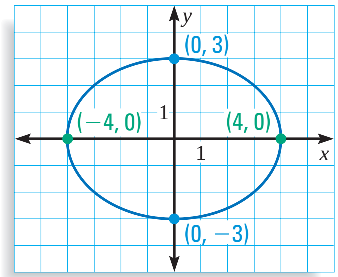
Draw the ellipse given by $9x^2 + 16y^2 = 144$. Identify the foci.

SOLUTION

First rewrite the equation in standard form.

$$\frac{9x^2}{144} + \frac{16y^2}{144} = \frac{144}{144} \qquad \textbf{Divide each side by 144.}$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \qquad \textbf{Simplify.}$$



Because the denominator of the x^2-term is greater than that of the y^2-term, the major axis is horizontal. So, $a = 4$ and $b = 3$. Plot the vertices and co-vertices. Then draw the ellipse that passes through these four points.

The foci are at $(c, 0)$ and $(-c, 0)$. To find the value of c, use the equation $c^2 = a^2 - b^2$.

$$c^2 = 4^2 - 3^2 = 16 - 9 = 7$$

$$c = \sqrt{7}$$

▶ The foci are at $\left(\sqrt{7}, 0\right)$ and $\left(-\sqrt{7}, 0\right)$.

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

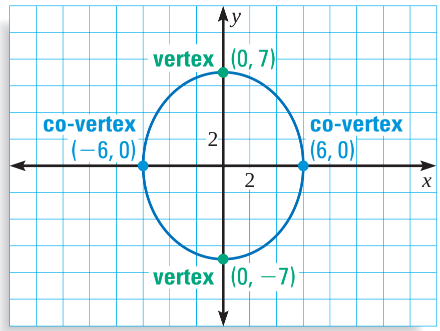
EXAMPLE 2 *Writing Equations of Ellipses*

Write an equation of the ellipse with the given characteristics and center at $(0, 0)$.

a. Vertex: $(0, 7)$
Co-vertex: $(-6, 0)$

b. Vertex: $(-4, 0)$
Focus: $(2, 0)$

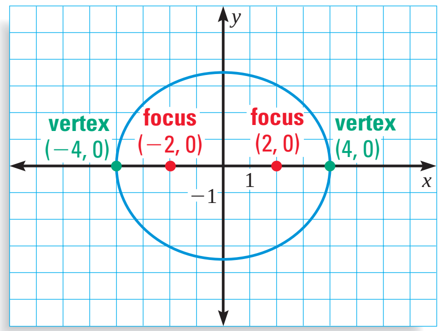
SOLUTION



In each case, you may wish to draw the ellipse so that you have something to check your final equation against.

a. Because the vertex is on the y-axis and the co-vertex is on the x-axis, the major axis is vertical with $a = 7$ and $b = 6$.

▶ An equation is $\dfrac{x^2}{36} + \dfrac{y^2}{49} = 1$.



b. Because the vertex and focus are points on a horizontal line, the major axis is horizontal with $a = 4$ and $c = 2$. To find b, use the equation $c^2 = a^2 - b^2$.

$$2^2 = 4^2 - b^2$$

$$b^2 = 16 - 4 = 12$$

$$b = 2\sqrt{3}$$

▶ An equation is $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$.

GOAL 2 USING ELLIPSES IN REAL LIFE

Both man-made objects, such as The Ellipse at the White House, and natural phenomena, such as the orbits of planets, involve ellipses.

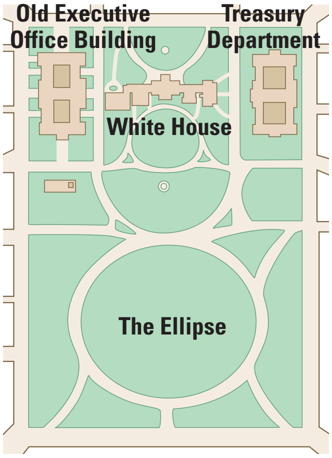
REAL LIFE
Landscaping

EXAMPLE 3 *Finding the Area of an Ellipse*

A portion of the White House lawn is called The Ellipse. It is 1060 feet long and 890 feet wide.

a. Write an equation of The Ellipse.

b. The area of an ellipse is $A = \pi ab$. What is the area of The Ellipse at the White House?



SOLUTION

a. The major axis is horizontal with

$$a = \frac{1060}{2} = 530 \text{ and } b = \frac{890}{2} = 445.$$

▶ An equation is $\dfrac{x^2}{530^2} + \dfrac{y^2}{445^2} = 1.$

b. The area is $A = \pi(530)(445) \approx 741{,}000$ square feet.

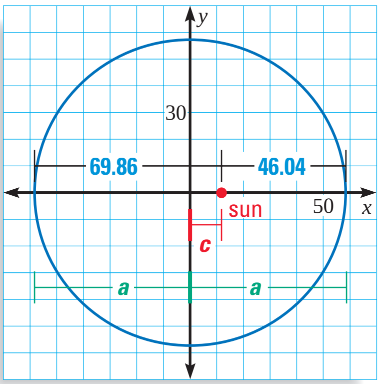
REAL LIFE
Astronomy

EXAMPLE 4 *Modeling with an Ellipse*

In its elliptical orbit, Mercury ranges from 46.04 million kilometers to 69.86 million kilometers from the center of the sun. The center of the sun is a focus of the orbit. Write an equation of the orbit.

SOLUTION

Using the diagram shown, you can write a system of linear equations involving a and c.



$$a - c = 46.04$$

$$a + c = 69.86$$

Adding the two equations gives $2a = 115.9$, so $a = 57.95$. Substituting this a-value into the second equation gives $57.95 + c = 69.86$, so $c = 11.91$.

From the relationship $c^2 = a^2 - b^2$, you can conclude the following:

$$b = \sqrt{a^2 - c^2}$$

$$= \sqrt{(57.95)^2 - (11.91)^2}$$

$$\approx 56.71$$

▶ An equation of the elliptical orbit is $\dfrac{x^2}{(57.95)^2} + \dfrac{y^2}{(56.71)^2} = 1$ where x and y are in millions of kilometers.

GUIDED PRACTICE

Vocabulary Check ✓

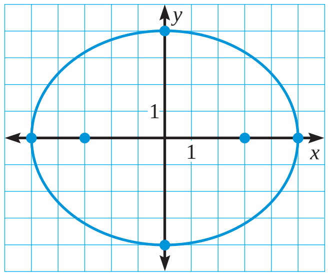
1. Complete each statement using the ellipse shown.

a. The points $(-5, 0)$ and $(5, 0)$ are called the ___?___.

b. The points $(0, -4)$ and $(0, 4)$ are called the ___?___.

c. The points $(-3, 0)$ and $(3, 0)$ are called the ___?___.

d. The segment with endpoints $(-5, 0)$ and $(5, 0)$ is called the ___?___.



Ex. 1

Concept Check ✓

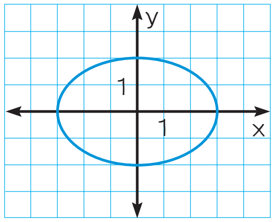
2. How can you tell from the equation of an ellipse whether the major axis is horizontal or vertical?

3. Explain how to find the foci of an ellipse given the coordinates of its vertices and co-vertices.

4. ERROR ANALYSIS A student was asked to write an equation of the ellipse shown at the right.

The student wrote the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

What did the student do wrong? What is the correct equation?



Ex. 4

Skill Check ✓

Write an equation of the ellipse with the given characteristics and center at (0, 0).

5. Vertex: $(0, 5)$
Co-vertex: $(-4, 0)$

6. Vertex: $(9, 0)$
Co-vertex: $(0, 2)$

7. Vertex: $(-7, 0)$
Focus: $(-2\sqrt{10}, 0)$

8. Vertex: $(0, 13)$
Focus: $(0, -5)$

9. Co-vertex: $(\sqrt{91}, 0)$
Focus: $(0, 3)$

10. Co-vertex: $(0, \sqrt{33})$
Focus: $(4, 0)$

Draw the ellipse.

11. $\frac{x^2}{49} + \frac{y^2}{25} = 1$

12. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

13. $\frac{x^2}{30} + \frac{y^2}{4} = 1$

14. $\frac{x^2}{64} + \frac{y^2}{45} = 1$

15. $75x^2 + 36y^2 = 2700$

16. $81x^2 + 63y^2 = 5103$

17. GARDEN An elliptical garden is 10 feet long and 6 feet wide. Write an equation for the garden. Then graph the equation. Label the vertices, co-vertices, and foci. Assume that the major axis of the garden is horizontal.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 954.

IDENTIFYING PARTS **Write the equation in standard form (if not already). Then identify the vertices, co-vertices, and foci of the ellipse.**

18. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

19. $\frac{x^2}{121} + \frac{y^2}{100} = 1$

20. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

21. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

22. $\frac{x^2}{12} + \frac{y^2}{36} = 1$

23. $\frac{x^2}{28} + \frac{y^2}{20} = 1$

24. $16x^2 + y^2 = 16$

25. $49x^2 + 4y^2 = 196$

26. $9x^2 + 100y^2 = 900$

27. $x^2 + 10y^2 = 10$

28. $10x^2 + 25y^2 = 250$

29. $25x^2 + 15y^2 = 375$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 18–50
Example 2: Exs. 51–68
Examples 3, 4: Exs. 69–75

GRAPHING **Graph the equation. Then identify the vertices, co-vertices, and foci of the ellipse.**

30. $\frac{x^2}{16} + \frac{y^2}{36} = 1$ **31.** $\frac{x^2}{4} + \frac{y^2}{49} = 1$ **32.** $\frac{x^2}{36} + \frac{y^2}{64} = 1$

33. $\frac{x^2}{49} + \frac{y^2}{144} = 1$ **34.** $\frac{x^2}{196} + \frac{y^2}{100} = 1$ **35.** $\frac{x^2}{256} + \frac{y^2}{36} = 1$

36. $\frac{x^2}{225} + \frac{y^2}{81} = 1$ **37.** $\frac{x^2}{121} + \frac{y^2}{169} = 1$ **38.** $\frac{x^2}{144} + \frac{y^2}{400} = 1$

39. $\frac{x^2}{49} + \frac{y^2}{64} = 1$ **40.** $\frac{x^2}{4} + y^2 = 100$ **41.** $\frac{x^2}{4} + \frac{y^2}{25} = \frac{1}{4}$

GRAPHING **In Exercises 42–50, the equations of parabolas, circles, and ellipses are given. Graph the equation.**

42. $x^2 + y^2 = 33^2$ **43.** $64x^2 + 25y^2 = 1600$ **44.** $24y + x^2 = 0$

45. $72x^2 = 144y$ **46.** $24x^2 + 24y^2 = 96$ **47.** $\frac{x^2}{81} + \frac{4y}{9} = 1$

48. $\frac{3x^2}{12} + \frac{5y^2}{500} = 1$ **49.** $\frac{x^2}{36} + \frac{y^2}{36} = 4$ **50.** $5x^2 + 9y^2 = 45$

WRITING EQUATIONS **Write an equation of the ellipse with the given characteristics and center at (0, 0).**

51. Vertex: $(0, 6)$
Co-vertex: $(5, 0)$

52. Vertex: $(0, 6)$
Co-vertex: $(-2, 0)$

53. Vertex: $(-4, 0)$
Co-vertex: $(0, 3)$

54. Vertex: $(0, -7)$
Co-vertex: $(-1, 0)$

55. Vertex: $(9, 0)$
Co-vertex: $(0, -8)$

56. Vertex: $(10, 0)$
Co-vertex: $(0, 4)$

57. Vertex: $(0, 7)$
Focus: $(0, 3)$

58. Vertex: $(-5, 0)$
Focus: $(2\sqrt{6}, 0)$

59. Vertex: $(0, 8)$
Focus: $(0, -4\sqrt{3})$

60. Vertex: $(15, 0)$
Focus: $(12, 0)$

61. Vertex: $(5, 0)$
Focus: $(-3, 0)$

62. Vertex: $(0, -30)$
Focus: $(0, 20)$

63. Co-vertex: $(\sqrt{55}, 0)$
Focus: $(0, -3)$

64. Co-vertex: $(0, -\sqrt{3})$
Focus: $(-1, 0)$

65. Co-vertex: $(-2\sqrt{10}, 0)$
Focus: $(0, 9)$

66. Co-vertex: $(0, -3\sqrt{3})$
Focus: $(3, 0)$

67. Co-vertex: $(5\sqrt{11}, 0)$
Focus: $(0, -7)$

68. Co-vertex: $(0, -\sqrt{77})$
Focus: $(-2, 0)$

WHISPERING GALLERY **In Exercises 69–71, use the following information.**
Statuary Hall is an elliptical room in the United States Capitol in Washington, D.C. The room is also called the Whispering Gallery because a person standing at one focus of the room can hear even a whisper spoken by a person standing at the other focus. This occurs because any sound that is emitted from one focus of an ellipse will reflect off the side of the ellipse to the other focus. Statuary Hall is 46 feet wide and 97 feet long.

69. Find an equation that models the shape of the room.

70. How far apart are the two foci?

71. What is the area of the floor of the room?

72. **SPACE EXPLORATION** The first artificial satellite to orbit Earth was Sputnik I, launched by the Soviet Union in 1957. The orbit was an ellipse with Earth's center as one focus. The orbit's highest point above Earth's surface was 583 miles, and its lowest point was 132 miles. Find an equation of the orbit. (Use 4000 miles as the radius of Earth.) Graph your equation.

AUSTRALIAN FOOTBALL **In Exercises 73–75, use the information below.** Australian football is played on an elliptical field. The official rules state that the field must be between 135 and 185 meters long and between 110 and 155 meters wide. ▶ Source: The Australian News Network

73. Write an equation for the largest allowable playing field.

74. Write an equation for the smallest allowable playing field.

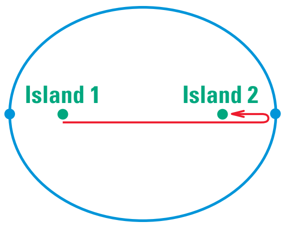
75. Write an inequality that describes the possible areas of an Australian football field.

Test Preparation

76. **MULTI-STEP PROBLEM** A tour boat travels between two islands that are 12 miles apart. For a trip between the islands, there is enough fuel for a 20-mile tour.

a. *Writing* The region in which the boat can travel is bounded by an ellipse. Explain why this is so.

b. Let (0, 0) represent the center of the ellipse. Find the coordinates of each island.



c. Suppose the boat travels from one island, straight past the other island to the vertex of the ellipse, and back to the second island. How many miles does the boat travel? Use your answer to find the coordinates of the vertex.

d. Use your answers to parts (b) and (c) to write an equation for the ellipse that bounds the region the boat can travel in.

★ **Challenge**

77. **LOGICAL REASONING** Show that $c^2 = a^2 - b^2$ for any ellipse given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci at $(c, 0)$ and $(-c, 0)$.

MIXED REVIEW

RATIONAL EXPONENTS **Evaluate the expression without using a calculator.** (Review 7.1)

78. $125^{2/3}$ **79.** $-8^{5/3}$ **80.** $4^{5/2}$ **81.** $27^{-2/6}$

82. $4^{7/2}$ **83.** $81^{3/4}$ **84.** $64^{-2/3}$ **85.** $32^{4/5}$

INVERSE VARIATION **The variables *x* and *y* vary inversely. Use the given values to write an equation relating *x* and *y*.** (Review 9.1)

86. $x = 3, y = -2$ **87.** $x = 4, y = 6$ **88.** $x = 5, y = 1$

89. $x = 8, y = 9$ **90.** $x = 9, y = 2$ **91.** $x = 0.5, y = 24$

GRAPHING **Graph the function. State the domain and range.** (Review 9.2 for 10.5)

92. $f(x) = \frac{9}{x}$ **93.** $f(x) = -\frac{9}{x}$ **94.** $f(x) = \frac{12}{x}$

95. $f(x) = \frac{24}{x}$ **96.** $f(x) = \frac{10}{x - 2}$ **97.** $f(x) = \frac{4}{x + 3}$