10.3

Circles

What *you should learn*

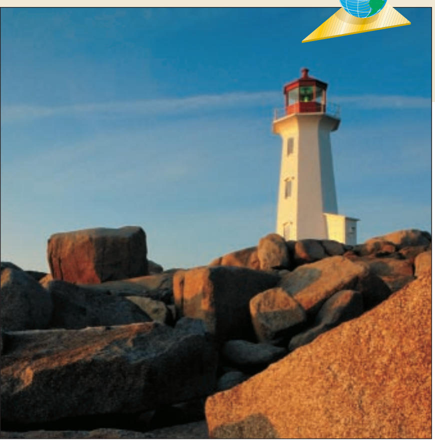
GOAL 1 Graph and write equations of circles.

GOAL 2 Use circles to solve **real-life** problems, such as determining whether you are affected by an earthquake in **Ex. 81**.

Why *you should learn it*

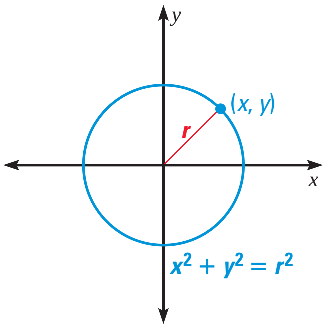
▼ To model **real-life** situations with circular models, such as the region lit by a lighthouse beam in **Example 4**.

REAL LIFE



GOAL 1 GRAPHING AND WRITING EQUATIONS OF CIRCLES

A **circle** is the set of all points (x, y) that are equidistant from a fixed point, called the **center** of the circle. The distance r between the center and any point (x, y) on the circle is the **radius**.



The distance formula can be used to obtain an equation of the circle whose center is the origin and whose radius is r. Because the distance between any point (x, y) on the circle and the center $(0, 0)$ is r, you can write the following.

$\sqrt{(x-0)^2 + (y-0)^2} = r$ **Distance formula**

$(x-0)^2 + (y-0)^2 = r^2$ **Square both sides.**

$x^2 + y^2 = r^2$ **Simplify.**

STANDARD EQUATION OF A CIRCLE (CENTER AT ORIGIN)

The **standard form of the equation of a circle** with center at (0, 0) and radius r is as follows:

$$x^2 + y^2 = r^2$$

EXAMPLE A circle with center at (0, 0) and radius 3 has equation $x^2 + y^2 = 9$.

EXAMPLE 1 *Graphing an Equation of a Circle*

Draw the circle given by $y^2 = 25 - x^2$.

SOLUTION

Write the equation in standard form.

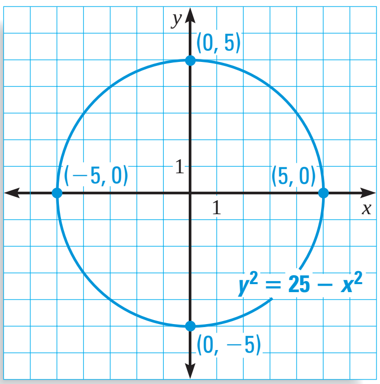
$y^2 = 25 - x^2$ **Original equation**

$x^2 + y^2 = 25$ **Add x^2 to each side.**

In this form you can see that the graph is a circle whose center is the origin and whose radius is $r = \sqrt{25} = 5$.

Plot several points that are 5 units from the origin. The points $(0, 5)$, $(5, 0)$, $(0, -5)$, and $(-5, 0)$ are most convenient.

Draw a circle that passes through the four points.



EXAMPLE 2 *Writing an Equation of a Circle*

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

The point (1, 4) is on a circle whose center is the origin. Write the standard form of the equation of the circle.

SOLUTION

Because the point (1, 4) is on the circle, the radius of the circle must be the distance between the center and the point (1, 4).

$$r = \sqrt{(1-0)^2 + (4-0)^2} \qquad \text{Use the distance formula.}$$

$$= \sqrt{1 + 16} \qquad \text{Simplify.}$$

$$= \sqrt{17}$$

Knowing that the radius is $\sqrt{17}$, you can use the standard form to find an equation of the circle.

$$x^2 + y^2 = r^2 \qquad \text{Standard form}$$

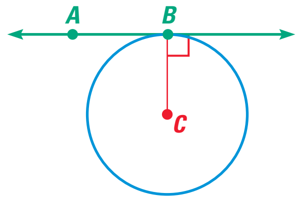
$$x^2 + y^2 = (\sqrt{17})^2 \qquad \text{Substitute } \sqrt{17} \text{ for } r.$$

$$x^2 + y^2 = 17 \qquad \text{Simplify.}$$

STUDENT HELP

Study Tip
In mathematics the term radius is used in two ways. As defined on the previous page, it is the distance from the center of a circle to a point on the circle. It can also refer to the line segment that connects the center to a point on the circle.

A theorem in geometry states that a line tangent to a circle is perpendicular to the circle's radius at the point of tangency. In the diagram, $\overleftrightarrow{AB}$ is tangent to the circle with center C at the point of tangency B, so $\overleftrightarrow{AB} \perp \overline{BC}$. This property of circles is used in the next example.



EXAMPLE 3 *Finding a Tangent Line*

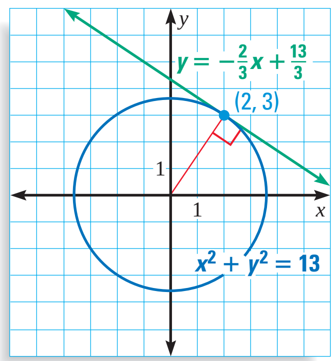
Write an equation of the line that is tangent to the circle $x^2 + y^2 = 13$ at (2, 3).

SOLUTION

The slope of the radius through the point (2, 3) is:

$$m = \frac{3-0}{2-0} = \frac{3}{2}$$

Because the tangent line at (2, 3) is perpendicular to this radius, its slope must be the negative reciprocal of $\frac{3}{2}$, or $-\frac{2}{3}$. So, an equation of the tangent line is as follows.



$$y - 3 = -\frac{2}{3}(x - 2) \qquad \text{Point-slope form}$$

$$y - 3 = -\frac{2}{3}x + \frac{4}{3} \qquad \text{Distributive property}$$

$$y = -\frac{2}{3}x + \frac{13}{3} \qquad \text{Add 3 to each side.}$$

An equation of the tangent line is $y = -\frac{2}{3}x + \frac{13}{3}$.

FOCUS ON APPLICATIONS

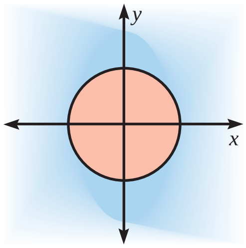
REAL LIFE **THE PHAROS OF ALEXANDRIA** was a lighthouse built in Egypt in about 280 B.C. One of the Seven Wonders of the World, it was said to be over 440 feet tall. It stood for nearly 1400 years.

GOAL 2 USING CIRCLES IN REAL LIFE

The regions inside and outside the circle $x^2 + y^2 = r^2$ can be described by inequalities.

Region inside circle: $x^2 + y^2 < r^2$

Region outside circle: $x^2 + y^2 > r^2$



EXAMPLE 4 *Using a Circular Model*

OCEAN NAVIGATION The beam of a lighthouse can be seen for up to 20 miles. You are on a ship that is 10 miles east and 16 miles north of the lighthouse.

a. Write an inequality to describe the region lit by the lighthouse beam.

b. Can you see the lighthouse beam?

SOLUTION

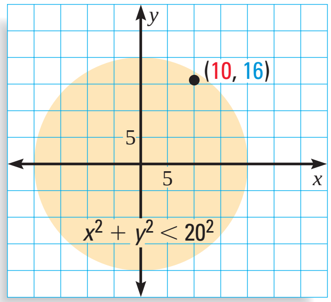
a. As shown at the right the lighthouse beam can be seen from all points that satisfy this inequality:

$$x^2 + y^2 < 20^2$$

b. Substitute the coordinates of the ship into the inequality you wrote in part (a).

$x^2 + y^2 < 20^2$	**Inequality from part (a)**
$10^2 + 16^2 \overset{?}{<} 20^2$	**Substitute for *x* and *y*.**
$100 + 256 \overset{?}{<} 400$	**Simplify.**
$356 < 400$ ✓	**The inequality is true.**

▶ You can see the beam from the ship.



In the diagram above, the origin represents the lighthouse and the positive *y*-axis represents north.

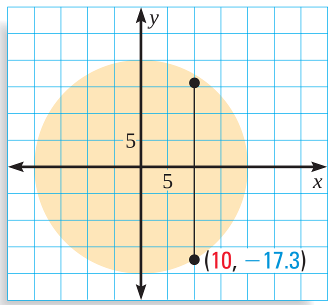
EXAMPLE 5 *Using a Circular Model*

OCEAN NAVIGATION Your ship in Example 4 is traveling due south. For how many more miles will you be able to see the beam?

SOLUTION

When the ship exits the region lit by the beam, it will be at a point on the circle $x^2 + y^2 = 20^2$. Furthermore, its x-coordinate will be 10 and its y-coordinate will be negative. Find the point $(10, y)$ where $y < 0$ on the circle $x^2 + y^2 = 20^2$.

$x^2 + y^2 = 20^2$	**Equation for the boundary**
$10^2 + y^2 = 20^2$	**Substitute 10 for *x*.**
$y = \pm\sqrt{300} \approx \pm 17.3$	**Solve for *y*.**



▶ Since $y < 0$, $y \approx -17.3$. The beam will be in view as the ship travels from (10, 16) to (10, −17.3), a distance of $|16 - (-17.3)| = 33.3$ miles.

GUIDED PRACTICE

Vocabulary Check ✓

1. State the definition of a circle.

Concept Check ✓

2. **LOGICAL REASONING** Tell whether the following statement is *always true*, *sometimes true*, or *never true*: For a given circle and a given x-coordinate, there are two points on the circle with that x-coordinate.

3. How is the slope of a line tangent to a circle related to the slope of the radius at the point of tangency?

4. **ERROR ANALYSIS** A student was asked to write an equation of a circle with its center at the origin and a radius of 4. The student wrote the following equation:

$$x^2 + y^2 = 4$$

What did the student do wrong? Write the correct equation.

Skill Check ✓

Write the standard form of the equation of the circle that passes through the given point and whose center is the origin.

5. $(4, 0)$ **6.** $(0, -2)$ **7.** $(-8, 6)$ **8.** $(-5, -12)$

9. $(6, -9)$ **10.** $(3, 1)$ **11.** $(-5, -5)$ **12.** $(-2, 4)$

Graph the equation. Give the radius of the circle.

13. $x^2 + y^2 = 36$ **14.** $x^2 + y^2 = 81$ **15.** $x^2 + y^2 = 32$

16. $x^2 + y^2 = 12$ **17.** $36x^2 + 36y^2 = 144$ **18.** $9x^2 + 9y^2 = 162$

19. **SHOT PUT** A person throws a shot put from a circle that has a diameter of 7 feet. Write the standard form of the equation of the shot put circle if the center is the origin.

PRACTICE AND APPLICATIONS

STUDENT HELP

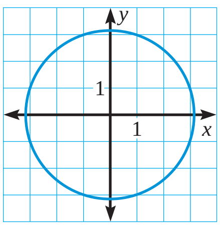
Extra Practice to help you master skills is on p. 954.

MATCHING GRAPHS **Match the equation with its graph.**

20. $x^2 + y^2 = 16$ **21.** $x^2 + y^2 = 5$ **22.** $x^2 + y^2 = 4$

23. $x^2 + y^2 = 25$ **24.** $x^2 + y^2 = 100$ **25.** $x^2 + y^2 = 10$

A.



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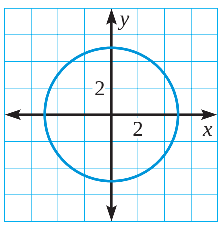

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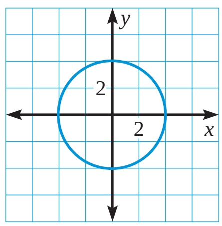

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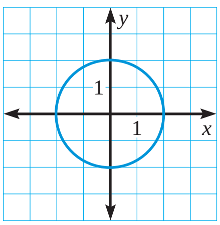
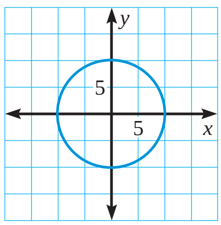
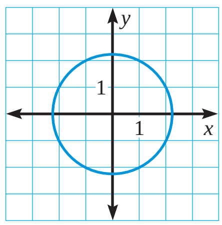


E.



F.



STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 20–46
Example 2: Exs. 47–70
Example 3: Exs. 71–79
Examples 4, 5: Exs. 81–87

GRAPHING Graph the equation. Give the radius of the circle.

26. $x^2 + y^2 = 1$ **27.** $x^2 + y^2 = 49$ **28.** $x^2 + y^2 = 64$

29. $x^2 + y^2 = 20$ **30.** $x^2 + y^2 = 8$ **31.** $x^2 + y^2 = 10$

32. $x^2 + y^2 = 3$ **33.** $5x^2 + 5y^2 = 80$ **34.** $24x^2 + 24y^2 = 96$

35. $8x^2 + 8y^2 = 192$ **36.** $9x^2 + 9y^2 = 135$ **37.** $4x^2 + 4y^2 = 52$

GRAPHING In Exercises 38–46, the equations of both circles and parabolas are given. Graph the equation.

38. $x^2 + y^2 = 11$ **39.** $x^2 + y^2 = 1$ **40.** $x^2 + y = 0$

41. $\frac{1}{4}x^2 + \frac{1}{4}y^2 = 16$ **42.** $4x^2 + y = 0$ **43.** $9x^2 + 9y^2 = 441$

44. $-2x + 9y^2 = 0$ **45.** $\frac{3}{8}x^2 + \frac{3}{8}y^2 = 6$ **46.** $x^2 + 12y = 0$

WRITING EQUATIONS Write the standard form of the equation of the circle with the given radius and whose center is the origin.

47. 3 **48.** 9 **49.** 6 **50.** 11

51. $\sqrt{7}$ **52.** $\sqrt{30}$ **53.** $\sqrt{11}$ **54.** $\sqrt{21}$

55. $5\sqrt{6}$ **56.** $4\sqrt{5}$ **57.** $2\sqrt{7}$ **58.** $3\sqrt{3}$

WRITING EQUATIONS Write the standard form of the equation of the circle that passes through the given point and whose center is the origin.

59. $(0, -10)$ **60.** $(8, 0)$ **61.** $(-3, -4)$ **62.** $(-4, -1)$

63. $(5, -3)$ **64.** $(-6, 4)$ **65.** $(-6, 1)$ **66.** $(-1, -9)$

67. $(7, -4)$ **68.** $(10, 2)$ **69.** $(5, 8)$ **70.** $(2, -12)$

FINDING TANGENT LINES The equation of a circle and a point on the circle is given. Write an equation of the line that is tangent to the circle at that point.

71. $x^2 + y^2 = 10; (1, 3)$ **72.** $x^2 + y^2 = 5; (2, 1)$

73. $x^2 + y^2 = 41; (-4, -5)$ **74.** $x^2 + y^2 = 145; (12, 1)$

75. $x^2 + y^2 = 65; (-8, 1)$ **76.** $x^2 + y^2 = 40; (-2, 6)$

77. $x^2 + y^2 = 244; (-10, -12)$ **78.** $x^2 + y^2 = \frac{257}{4}; \left(\frac{1}{2}, -8\right)$

79. CRITICAL THINKING Look back at Example 3. Find an equation of the line that is tangent to the circle at the point $(2, -3)$. Describe how the line is geometrically related to the line found in Example 3.

80. *Writing* Describe how the equation of a circle is related to the Pythagorean theorem. Include a diagram to illustrate the relationship.

81. EARTHQUAKES Suppose an earthquake can be felt up to 80 miles from its epicenter. You are located at a point 60 miles west and 45 miles south of the epicenter. Do you feel the earthquake? If so, how many miles south would you have to travel to be out of the range of the earthquake?

82. DESERT IRRIGATION A circular field has an area of about 2,400,000 square yards. Write an equation that represents the boundary of the field. Let $(0, 0)$ represent the center of the field.

FOCUS ON APPLICATIONS

REAL LIFE **IRRIGATION** This irrigation project in Colorado enables farmers to raise crops in the desert. Water from deep wells is pumped to sprinklers that rotate, forming circular patterns.

FOCUS ON CAREERS

AIR TRAFFIC CONTROLLER

Air traffic controllers are responsible for making sure airplanes fly a safe distance away from one another. They also help keep flights on schedule.

CAREER LINK
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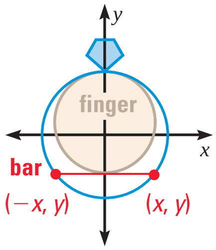
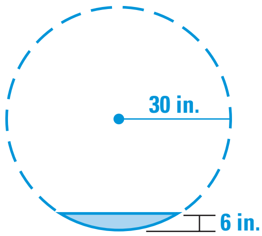
83. RESIZING A RING One way to resize a ring is to fit a bar into the ring, as shown. Suppose a ring that is 20 millimeters in diameter has to be resized to fit a finger 16 millimeters in diameter. What is the length of the bar that should be inserted in order to make the ring fit the finger? (*Hint:* Write an equation of the ring, assuming it is centered at the origin. Determine what the y-coordinate of the bar must be and then substitute this coordinate into the equation to find x.)



84. LIFEGUARD You are a lifeguard at a pond. The pond is a circle with a diameter of 360 feet. You want to rope off a section of the pond for swimming. If you want the rope to form a chord of the circle and have a maximum distance of 45 feet from shore, approximately how long will you need the rope to be?

85. PHYSICAL THERAPY A tilt-board is a physical therapy device that a person rocks back and forth on. Suppose the ends of a tilt-board are part of a circle with a radius of 30 inches. If the tilt-board has a depth of 6 inches, how wide is it?

▶ Source: Steps to Follow



AIR TRAFFIC CONTROL **In Exercises 86 and 87, use the following information.** An air traffic control tower can detect airplanes up to 50 miles away. You are in an airplane 42 miles east and 43 miles south of the control tower.

86. Write an inequality that describes the region in which planes can be detected by the control tower. Can the control tower detect your plane on its radar?

87. Suppose a jet is 35 miles west and 66 miles north of the control tower and is traveling due south at a speed of 500 miles per hour. After how many minutes will the jet appear on the control tower's radar?

Test Preparation

88. MULTIPLE CHOICE What is the equation of the line that is tangent to the circle $x^2 + y^2 = 53$ at the point $(7, 2)$?

(A) $y = -\frac{7}{2}x + \frac{45}{2}$ (B) $y = -\frac{7}{2}x + \frac{53}{2}$

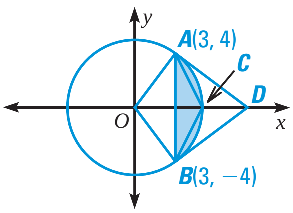
(C) $y = \frac{7}{2}x - \frac{45}{2}$ (D) $y = -\frac{7}{2}x - \frac{45}{2}$

89. MULTIPLE CHOICE Suppose a signal from a television transmitter tower can be received up to 150 miles away. The following points represent the locations of houses near the transmitter tower with the origin representing the tower. Which point is *not* within the range of the tower?

(A) (120, 20) (B) (40, 140) (C) (105, 120) (D) (10, 148)

★ Challenge

90. ESTIMATING AREA The *segment* of a circle is the region bounded by a chord and an arc. Estimate the area of the shaded segment by finding the area of $\triangle ABC$ and the area of $\triangle ABD$, given that $\overline{AD}$ and $\overline{BD}$ are tangent to the circle.



MIXED REVIEW

SOLVING SYSTEMS **Solve the system using any algebraic method.** **(Review 3.2)**

91. $x - 9y = 25$
$6x - 5y = 3$

92. $9x - y = 8$
$3x + 10y = -49$

93. $2x - 3y = 2$
$-7x + 4y = 6$

94. $8x - 5y = 4$
$2x + y = 1$

95. $-x + 5y = 3$
$4x - 9y = 10$

96. $-9x + 4y = 15$
$3x + 2y = 5$

COMPOSITION OF FUNCTIONS **Find $f(g(x))$ and $g(f(x))$.** **(Review 7.3)**

97. $f(x) = x + 1$ and $g(x) = 2x$

98. $f(x) = 4x + 1$ and $g(x) = x - 5$

99. $f(x) = -x^2 - 1$ and $g(x) = x + 5$

100. $f(x) = x^2 - 7$ and $g(x) = 3x + 1$

GRAPHING FUNCTIONS **Graph the function.** **(Review 8.1)**

101. $y = \frac{1}{4} \cdot 5^x$

102. $y = -\left(\frac{5}{3}\right)^x$

103. $y = 4 \cdot 3^{x-1} - 7$

104. $y = 3 \cdot 2^{x-4}$

105. $y = 2^{x+3} - 1$

106. $y = \frac{1}{4} \cdot 8^{x+1}$

BABYSITTING **In Exercises 107 and 108, use the following information.** In June you babysit 35 hours for the Johnsons and 52 hours for the Martins. In July you babysit 112 hours for the Johnsons and 40 hours for the Martins. In August you babysit 95 hours for the Johnsons and 63 hours for the Martins. **(Review 4.1)**

107. Use a matrix to organize the information.

108. You charge \$6 per hour for babysitting. Using your matrix from Exercise 107, write a matrix that shows how much you earned over the summer vacation.

QUIZ 1

Self-Test for Lessons 10.1–10.3

Find the distance between the two points. Then find the midpoint of the line segment joining the two points. **(Lesson 10.1)**

1. (0, 0), (8, 6)

2. (3, 3), (−3, −3)

3. (−2, 7), (4, −10)

4. (3, −7), (−5, −9)

5. (8, 6), (−4, 4)

6. (−1, −13), (11, 15)

Draw the parabola. Identify the focus and directrix. **(Lesson 10.2)**

7. $y^2 = 6x$

8. $3y = x^2$

9. $-x^2 = 5y$

10. $-4y^2 = 6x$

11. $3x^2 = 7y$

12. $\frac{1}{2}x = 2y^2$

13. $x + \frac{1}{8}y^2 = 0$

14. $-x^2 - 12y = 0$

Write the standard form of the equation of the circle that passes through the given point and whose center is the origin. **(Lesson 10.3)**

15. (0, 3)

16. (−5, 0)

17. (4, 7)

18. (−2, −5)

19. (−1, 9)

20. (6, −3)

21. (6, −6)

22. (−7, 8)

23. **RADIO SIGNALS** The signals of a radio station can be received up to 65 miles away. Your house is 35 miles east and 56 miles south of the radio station. Can you receive the radio station's signals? Explain. **(Lesson 10.3)**