7.3 Solving Linear Systems by Linear Combinations

What *you should learn*

GOAL 1 Use linear combinations to solve a system of linear equations.

GOAL 2 Model a **real-life** problem using a system of linear equations, such as the mixture problem in **Example 4**.

Why *you should learn it*

▼ To solve **real-life** problems such as finding the speed of the current in a river in **Ex. 48**.

REAL LIFE

GOAL 1 USING LINEAR COMBINATIONS

Sometimes when you want to solve a linear system, it is not easy to isolate one of the variables. In that case, you can solve the system by *linear combinations*. A **linear combination** of two equations is an equation obtained by adding one of the equations (or a multiple of one of the equations) to the other equation.

SOLVING A LINEAR SYSTEM BY LINEAR COMBINATIONS

STEP 1 Arrange the equations with like terms in columns.

STEP 2 Multiply one or both of the equations by a number to obtain coefficients that are opposites for one of the variables.

STEP 3 Add the equations from Step 2. Combining like terms will eliminate one variable. Solve for the remaining variable.

STEP 4 Substitute the value obtained in Step 3 into either of the original equations and solve for the other variable.

STEP 5 Check the solution in each of the original equations.

EXAMPLE 1 *Using Addition*

Solve the linear system. $4x + 3y = 16$ **Equation 1**
$2x - 3y = 8$ **Equation 2**

SOLUTION

1. The equations are already arranged.
2. The coefficients for y are already opposites.
3. Add the equations to get an equation in one variable.

$$\begin{array}{rl} 4x + 3y = 16 & \textbf{Write Equation 1.} \\ \underline{2x - 3y = 8} & \textbf{Write Equation 2.} \\ 6x = 24 & \textbf{Add equations.} \\ x = 4 & \textbf{Solve for } x. \end{array}$$

4. Substitute 4 for x in the first equation and solve for y.

$$\begin{array}{rl} 4(4) + 3y = 16 & \textbf{Substitute 4 for } x. \\ y = 0 & \textbf{Solve for } y. \end{array}$$

5. Check by substituting 4 for x and 0 for y in each of the original equations.

▶ The solution is (4, 0).

EXAMPLE 2 *Using Multiplication First*

Solve the linear system. $3x + 5y = 6$ **Equation 1**

$-4x + 2y = 5$ **Equation 2**

SOLUTION

The equations are already arranged. You can get the coefficients of x to be opposites by multiplying the first equation by 4 and the second equation by 3.

$$\begin{aligned} 3x + 5y &= 6 \quad \xrightarrow{\text{Multiply by 4.}} \quad & 12x + 20y &= 24 \\ -4x + 2y &= 5 \quad \xrightarrow{\text{Multiply by 3.}} \quad & -12x + 6y &= 15 \\ & & 26y &= 39 \quad \textbf{Add equations.} \\ & & y &= 1.5 \quad \textbf{Solve for } \boldsymbol{y}\textbf{.} \end{aligned}$$

Substitute 1.5 for y in the second equation and solve for x.

$$\begin{aligned} -4x + 2y &= 5 && \textbf{Write Equation 2.} \\ -4x + 2(1.5) &= 5 && \textbf{Substitute 1.5 for } \boldsymbol{y}\textbf{.} \\ -4x + 3 &= 5 && \textbf{Simplify.} \\ x &= -0.5 && \textbf{Solve for } \boldsymbol{x}\textbf{.} \end{aligned}$$

▶ The solution is $(-0.5, 1.5)$. Check this in the original equations.

STUDENT HELP

Study Tip
Sometimes you can add the original equations because the coefficients are already opposites as in Example 1. In Example 2 you need to multiply both equations by appropriate numbers first.

EXAMPLE 3 *Arranging Like Terms in Columns*

Solve the linear system. $3x + 2y = 8$ **Equation 1**

$2y = 12 - 5x$ **Equation 2**

SOLUTION

First arrange the equations.

$$\begin{aligned} 3x + 2y &= 8 && \textbf{Write Equation 1.} \\ 5x + 2y &= 12 && \textbf{Rearrange Equation 2.} \end{aligned}$$

You can get the coefficients of y to be opposites by multiplying the second equation by -1.

$$\begin{aligned} 3x + 2y &= 8 & & 3x + 2y &= 8 \\ 5x + 2y &= 12 \quad \xrightarrow{\text{Multiply by } -1\text{.}} \quad & & -5x - 2y &= -12 \\ & & & -2x &= -4 \quad \textbf{Add equations.} \\ & & & x &= 2 \quad \textbf{Solve for } \boldsymbol{x}\textbf{.} \end{aligned}$$

Substitute 2 for x into the first equation and solve for y.

$$\begin{aligned} 3x + 2y &= 8 && \textbf{Write Equation 1.} \\ 3(2) + 2y &= 8 && \textbf{Substitute 2 for } \boldsymbol{x}\textbf{.} \\ 6 + 2y &= 8 && \textbf{Simplify.} \\ y &= 1 && \textbf{Solve for } \boldsymbol{y}\textbf{.} \end{aligned}$$

▶ The solution is $(2, 1)$. Check this in the original equations.

STUDENT HELP

INTERNET

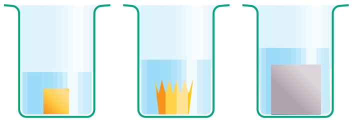
HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 MODELING A REAL-LIFE SITUATION

Legend has it that Archimedes used the relationship between the weight of an object and its volume to prove that his king's gold crown was not pure gold. If the crown was all gold, it would have had the same volume as an equal amount of gold.


Gold block has same weight as crown.


Silver block has same weight as crown.


Gold block, crown, and silver block displace different amounts of water.

EXAMPLE 4 *Writing and Using a Linear System*

GOLD AND SILVER A gold crown, suspected of containing some silver, was found to have a weight of 714 grams and a volume of 46 cubic centimeters. The density of gold is about 19 grams per cubic centimeter. The density of silver is about 10.5 grams per cubic centimeter. What percent of the crown is silver?

SOLUTION

VERBAL MODEL

$$\boxed{\text{Gold volume}} + \boxed{\text{Silver volume}} = \boxed{\text{Total volume}}$$

$$\boxed{\text{Gold density}} \cdot \boxed{\text{Gold volume}} + \boxed{\text{Silver density}} \cdot \boxed{\text{Silver volume}} = \boxed{\text{Total weight}}$$

LABELS

Volume of gold $= G$ (cubic centimeters)

Volume of silver $= S$ (cubic centimeters)

Total volume $= \mathbf{46}$ (cubic centimeters)

Density of gold $= \mathbf{19}$ (grams per cubic centimeter)

Density of silver $= \mathbf{10.5}$ (grams per cubic centimeter)

Total weight $= \mathbf{714}$ (grams)

ALGEBRAIC MODEL

$G + S = 46$ **Equation 1**

$19G + 10.5S = 714$ **Equation 2**

Use linear combinations to solve for S.

$-19G - 19S = -874$ **Multiply Equation 1 by −19.**

$19G + 10.5S = 714$ **Write Equation 2.**

$-8.5S = -160$ **Add equations.**

▶ The volume of silver is about 19 cm^3. The crown has a volume of 46 cm^3, so the crown is $\frac{19}{46} \approx 41\%$ silver by volume.

FOCUS ON PEOPLE

REAL LIFE **ARCHIMEDES** In the third century B.C., Archimedes supposedly used the relationship between the weight of an object and its volume to uncover a fraud in the manufacture of a gold crown.

INTERNET **APPLICATION LINK** www.mcdougallittell.com

GUIDED PRACTICE

Vocabulary Check ✓

1. When you use the linear combinations method to solve a linear system, what is the purpose of using multiplication as the first step?

Concept Check ✓

ERROR ANALYSIS **Find and describe the error. Then correctly solve the linear system by using linear combinations.**

2.

$$\begin{array}{rcr} x + y = 1 & \rightarrow & 10x + 10y = 10 \\ 5x + 4y = 14 & \rightarrow & -10x + 8y = -28 \\ \hline & & 18y = -18 \\ & & y = -1 \end{array}$$

3.

$$\begin{array}{rcr} 3x + y = 24 & \rightarrow & 9x + 3y = 24 \\ 7x - 3y = 8 & \rightarrow & 7x - 3y = 8 \\ \hline & & 2x \quad = 32 \\ & & x = 16 \end{array}$$

Skill Check ✓

Explain the steps you would use to solve the system of equations using linear combinations. Then solve the system.

4. $x + 3y = 6$
$x - 3y = 12$

5. $x - 3y = 0$
$x + 10y = 13$

6. $3x - 4y = 7$
$2x - y = 3$

7. $2y = -2 + 2x$
$2x + 3y = 12$

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 803.

USING ADDITION **Use linear combinations to solve the system of linear equations.**

8. $2x + y = 4$
$x - y = 2$

9. $a - b = 8$
$a + b = 20$

10. $y - 2x = 0$
$6y + 2x = 0$

11. $m + 3n = 2$
$-m + 2n = 3$

12. $p + 4q = 23$
$-p + q = 2$

13. $3v - 2w = 1$
$2v + 2w = 4$

14. $\frac{1}{2}g + h = 2$
$-g - h = 2$

15. $6.5x - 2.5y = 4.0$
$1.5x + 2.5y = 4.0$

USING MULTIPLICATION FIRST **Use linear combinations to solve the system of linear equations.**

16. $x - y = 0$
$-3x - y = 2$

17. $v - w = -5$
$v + 2w = 4$

18. $x + 3y = 3$
$x + 6y = 3$

19. $2g - 3h = 0$
$3g - 2h = 5$

20. $2p - q = 2$
$2p + 3q = 22$

21. $2a + 6z = 4$
$3a - 7z = 6$

22. $5e + 4f = 9$
$4e + 5f = 9$

23. $10m + 16n = 140$
$5m - 8n = 60$

24. $9x - 3z = 20$
$3x + 6z = 2$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 8–15
Example 2: Exs. 16–24, 31–42
Example 3: Exs. 25–42
Example 4: Ex. 43

ARRANGING LIKE TERMS **Use linear combinations to solve the system of linear equations.**

25. $x + 3y = 12$
$-3y + x = 30$

26. $3b + 2c = 46$
$5c + b = 11$

27. $y = x - 9$
$x + 8y = 0$

28. $2q = 7 - 5p$
$4p - 16 = q$

29. $2v = 150 - u$
$2u = 150 - v$

30. $0.1g - h + 4.3 = 0$
$3.6 = -0.2g + h$

STUDENT HELP

Look Back
For help with simplifying expressions, see p. 102.

SOLVING LINEAR SYSTEMS In Exercises 31–42, use linear combinations to solve the system of linear equations.

31. $x + 2y = 5$
$5x - y = 3$

32. $3p - 2 = -q$
$-q + 2p = 3$

33. $3g - 24 = -4h$
$-2 + 2h = g$

34. $t + r = 1$
$2r - t = 2$

35. $x + 1 - 3y = 0$
$2x = 7 - 3y$

36. $3a + 9b = 8b - a$
$5a - 10b = 4a - 9b + 5$

37. $2m - 4 = 4n$
$m - 2 = n$

38. $3y = -5x + 15$
$-y = -3x + 9$

39. $3j + 5k = 19$
$4j - 8k = -4$

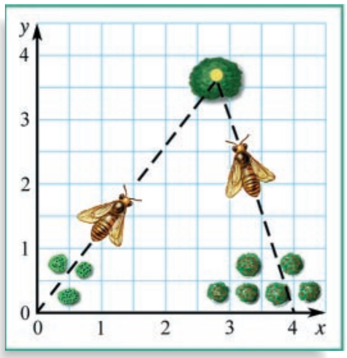
40. $1.5v - 6.5w = 3.5$
$0.5v + 2w = -3$

41. $5y - 20 = -4x$
$y = -\frac{5}{4}x + 4$

42. $9g - 7h = \frac{2}{3}$
$3g + h = \frac{1}{3}$

43. **WEIGHT OF GOLD** A gold and copper bracelet weighs 238 grams. The volume of the bracelet is 15 cubic centimeters. Gold weighs 19.3 grams per cubic centimeter, and copper weighs 9 grams per cubic centimeter. How many grams of copper are mixed with the gold?

44. **HONEY BEE PATHS** A farmer is tracking two wild honey bees in his field. He maps the first bee's path back to the hive on the line $y = \frac{9}{7}x$. The second bee's path follows the line $y = -3x + 12$. Their paths cross at the hive. At what coordinates will the farmer find the hive?



FOCUS ON APPLICATIONS

HONEY BEES In 1990, unusually cold weather and predators destroyed 90% of the wild honey bee colonies in the United States. If a colony is destroyed, farmers may need to rent bee hives to pollinate their crops.

APPLICATION LINK
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AIRPLANE SPEED In Exercises 45–47, use the following information. It took 3 hours for a plane, flying against the wind, to travel 900 miles from Alabama to Minnesota. The "ground speed" of the plane is 300 miles per hour. On the return trip, the flight took only 2 hours with a ground speed of 450 miles per hour. During both flights the speed and the direction of the wind were the same. The plane's speed decreases or increases because of the wind as the verbal model below shows.

$\boxed{\text{Speed in still air}} - \boxed{\text{Wind speed}} = \boxed{\text{Ground speed against wind}}$

$\boxed{\text{Speed in still air}} + \boxed{\text{Wind speed}} = \boxed{\text{Ground speed with wind}}$

45. Assign labels to the verbal model shown above. Use the labels to translate the verbal model into a system of linear equations.

46. Solve the linear system.

47. What was the speed of the plane in still air? What was the speed of the wind?

48. **STEAMBOAT SPEED** A steamboat went 8 miles upstream in 1 hour. The return trip took only 30 minutes. Assume that the speed of the current and the direction were constant during both parts of the trip. Find the speed of the boat in still water and the speed of the current.

SOLVING EFFICIENTLY **In Exercises 49–51, decide which variable to eliminate when using linear combinations to solve the system. Explain your thinking.**

49. $2x + 3y = 1$
$4x - 2y = 10$

50. $5y - 3x = 7$
$x + 3y = 7$

51. $\frac{1}{3}x + 6y = 6$
$-x + 3y = 3$

52. *Writing* Describe a general method for deciding which variable to eliminate when using linear combinations.

Test Preparation

QUANTITATIVE COMPARISON **In Exercises 53–55, solve the system. Then choose the statement below that is true about the solution of the system.**

Ⓐ The value of x is greater than the value of y.

Ⓑ The value of y is greater than the value of x.

Ⓒ The values of x and y are equal.

Ⓓ The relationship cannot be determined from the given information.

53. $x + y = 4$
$x - 2y = 1$

54. $x - 5y + 1 = 6$
$x = 12y$

55. $3x + 5y = -8$
$x - 2y = 1$

★ **Challenge**

56. SYSTEM OF THREE EQUATIONS Solve for x, y, and z in the system of equations. Explain each of your solution steps.

$3x + 2y + z = 42$
$2y + z + 12 = 3x$
$x - 3y = 0$

EXTRA CHALLENGE
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MIXED REVIEW

WRITING EQUATIONS **Write an equation of the line in slope-intercept form that passes through the two points, or passes through the point and has the given slope.** **(Review 5.2, 5.3)**

57. $(-2, -1), (4, 2)$

58. $(-2, 4), m = 3$

59. $(6, 5), (2, 1)$

60. $(5, 1), m = 5$

61. $(9, 3), m = -\frac{1}{3}$

62. $(4, -5), (-1, -3)$

CHECKING SOLUTIONS **Check whether each ordered pair is a solution of the inequality.** **(Review 6.5)**

63. $3x - 2y < 2$; $(1, 3), (2, 0)$

64. $5x + 4y \geq 6$; $(-2, 4), (5, 5)$

65. $5x + y > 5$; $(5, 5), (-5, -5)$

66. $12y - 3x \leq 3$; $(-2, 4), (1, -1)$

SOLVING SYSTEMS **Use substitution to solve the linear system.** **(Review 7.2 for 7.4)**

67. $-6x - 5y = 28$
$x - 2y = 1$

68. $m + 2n = 1$
$5m - 4n = -23$

69. $g - 5h = 20$
$4g + 3h = 34$

70. $p + 4q = -9$
$2p - 3q = 4$

71. $\frac{3}{5}b - a = 0$
$1 + b = 2a$

72. $d - e = 8$
$\frac{1}{5}d = e + 4$

QUIZ 1

Self-Test for Lessons 7.1–7.3

Graph and check to solve the linear system. (Lesson 7.1)

1. $3x + y = 5$
$-x + y = -7$

2. $\frac{1}{2}x + \frac{3}{4}y = 9$
$-2x + y = -4$

3. $x - 2y = 0$
$3x - y = 0$

Use substitution to solve the linear system. (Lesson 7.2)

4. $4x + 3y = 31$
$y = 2x + 7$

5. $-12x + y = 15$
$3x + 2y = 3$

6. $x + \frac{1}{2}y = 7$
$3x + 2y = 18$

Use linear combinations to solve the linear system. (Lesson 7.3)

7. $x + 7y = 12$
$3x - 5y = 10$

8. $3x - 5y = -4$
$-9x + 7y = 8$

9. $\frac{2}{3}x + \frac{1}{6}y = \frac{2}{3}$
$-y = 12 - 2x$

10. **COMPACT DISC SALE** A store is selling compact discs for \$10.50 and \$8.50. You buy 10 discs and spend a total of \$93. How many compact discs did you buy that cost \$10.50? that cost \$8.50? (Lessons 7.1–7.3)

MATH & History — Systems of Linear Equations

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THEN

THE FIRST KNOWN SYSTEM of linear equations appeared in a Chinese book about 2000 years ago. The problem below appeared in the book *Shu-shu chiu-chang* in 1247.

A storehouse has three kinds of stuff: cotton, floss silk, and raw silk. They take inventory of the materials and wish to cut out and make garments for the army. As for the cotton, if we use 8 rolls for 6 men, we have a shortage of 160 rolls; if we use 9 rolls for 7 men, there is a surplus of 560 rolls. . . . We wish to know the number of men [we can clothe] and the amounts of cotton [we will use]. . .

$$-x = -\frac{8y}{6} + 160$$
$$x = \frac{9y}{7} + 560$$
$$0 = \frac{9y}{7} - \frac{8y}{6} + 720$$

1. In the equations above, what does x represent? What does y represent?
2. Solve the linear system and interpret the solution.

NOW

CASH REGISTERS can keep track of inventory and notify a store's manager when inventory needs to be ordered.

c. 3000 B.C.

The first abacus

1879

Cash register is invented.

Now

Modern cash registers have scanners.