12.6 The Distance and Midpoint Formulas

What *you should learn*

GOAL 1 Find the distance between two points in a coordinate plane.

GOAL 2 Find the midpoint between two points in a coordinate plane.

Why *you should learn it*

▼ To find distances in **real-life** situations, such as the length of a soccer kick in **Example 3**.

REAL LIFE

GOAL 1 FINDING THE DISTANCE BETWEEN TWO POINTS

ACTIVITY Developing Concepts

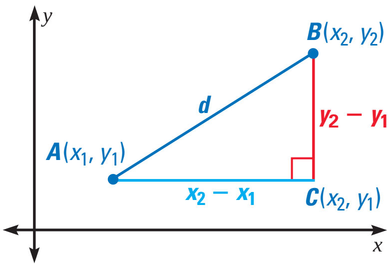
Investigating Distance

1. Plot $A(2, 1)$ and $B(6, 4)$ on graph paper. Then draw a right triangle that has $\overline{AB}$ as its hypotenuse.
2. Label the coordinates of the vertex C.
3. Find the lengths of the legs of $\triangle ABC$.
4. Use the Pythagorean theorem to find AB.
5. Check the distance by actual measurement.

The steps used in the investigation can be used to develop a general formula for the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Using the Pythagorean theorem, you can write the equation

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Solving this equation for d produces the **distance formula**.



THE DISTANCE FORMULA

The distance d between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 1 *Finding the Distance Between Two Points*

Find the distance between $(1, 4)$ and $(-2, 3)$.

SOLUTION To find the distance, use the distance formula.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **Write distance formula.**

$= \sqrt{(-2 - 1)^2 + (3 - 4)^2}$ **Substitute.**

$= \sqrt{10}$ **Simplify.**

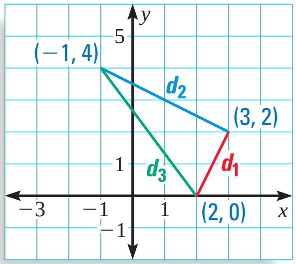
≈ 3.16 **Use a calculator.**

EXAMPLE 2 *Checking a Right Triangle*

Decide whether the points (3, 2), (2, 0), and (−1, 4) are vertices of a right triangle.

SOLUTION

Use the distance formula to find the lengths of the three sides.



$$d_1 = \sqrt{(3-2)^2 + (2-0)^2} = \sqrt{1+4} = \sqrt{5}$$

$$d_2 = \sqrt{[3-(-1)]^2 + (2-4)^2} = \sqrt{16+4} = \sqrt{20}$$

$$d_3 = \sqrt{[2-(-1)]^2 + (0-4)^2} = \sqrt{9+16} = \sqrt{25}$$

Next find the sum of the squares of the lengths of the two shorter sides.

$$d_1^{\,2} + d_2^{\,2} = (\sqrt{5})^2 + (\sqrt{20})^2 \quad \textbf{Substitute for } d_1 \textbf{ and } d_2.$$

$$= 5 + 20 \quad \textbf{Simplify.}$$

$$= 25 \quad \textbf{Add.}$$

The sum of the squares of the lengths of the shorter sides is 25, which is equal to the square of the length of the longest side, $(\sqrt{25})^2$.

▶ The given points are vertices of a right triangle.

. . . .

When you want to use the distance formula to find a distance in a real-life problem, the first step is to draw a diagram and assign coordinates to the points. This process is called *superimposing* a coordinate system on the diagram.

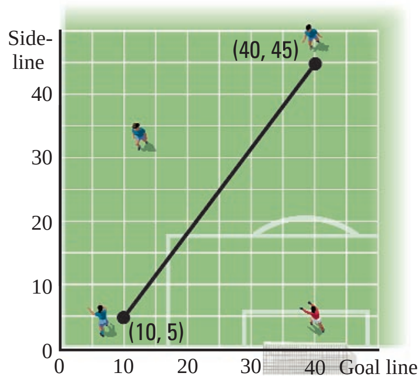
REAL LIFE
Soccer

EXAMPLE 3 *Applying the Distance Formula*

A player kicks a soccer ball from a position that is 10 yards from a sideline and 5 yards from a goal line. The ball lands at a position that is 45 yards from the same goal line and 40 yards from the same sideline. How far was the ball kicked?

SOLUTION

You can begin by superimposing a coordinate system on the soccer field. The ball is kicked from the point (10, 5). It lands at the point (40, 45). Use the distance formula.



$$d = \sqrt{(40-10)^2 + (45-5)^2}$$

$$= \sqrt{900 + 1600}$$

$$= \sqrt{2500}$$

$$= 50$$

▶ The ball was kicked 50 yards.

GOAL 2 FINDING THE MIDPOINT BETWEEN TWO POINTS

The **midpoint** of a line segment is the point on the segment that is equidistant from its end-points. The *midpoint between two points* is the midpoint of the line segment connecting them.

THE MIDPOINT FORMULA

The midpoint between (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

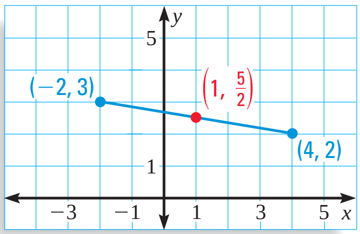
EXAMPLE 4 *Finding the Midpoint Between Two Points*

Find the midpoint between $(-2, 3)$ and $(4, 2)$. Use a graph to check the result.

SOLUTION

$$\left(\frac{-2 + 4}{2}, \frac{3 + 2}{2}\right) = \left(\frac{2}{2}, \frac{5}{2}\right) = \left(1, \frac{5}{2}\right)$$

▶ The midpoint is $\left(1, \frac{5}{2}\right)$.



✓**CHECK** From the graph, you can see that the point $\left(1, \frac{5}{2}\right)$ appears halfway between $(-2, 3)$ and $(4, 2)$. You can also use the distance formula to check that the distances from the midpoint to each given point are equal.

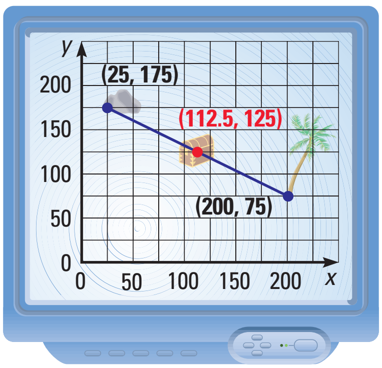
REAL LIFE Computers

EXAMPLE 5 *Applying the Midpoint Formula*

You are using computer software to design a video game. You want to place a buried treasure chest halfway between the center of the base of a palm tree and the corner of a large boulder. Find where you should place the treasure chest.

SOLUTION

Begin by assigning coordinates to the locations of the two landmarks. The center of the base of the palm tree is at (200, 75). The corner of the boulder is at (25, 175). Then use the midpoint formula to find the point that is halfway between the two landmarks.



$$\left(\frac{25 + 200}{2}, \frac{175 + 75}{2}\right) = \left(\frac{225}{2}, \frac{250}{2}\right)$$

$$= (112.5, 125)$$

▶ You should place the treasure chest at (112.5, 125).

GUIDED PRACTICE

Vocabulary Check ✓

1. What is meant by the *midpoint* between two points?

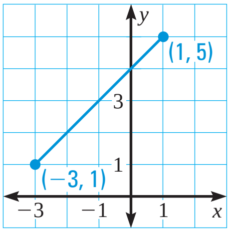
Concept Check ✓

2. Explain how you can use the Pythagorean theorem to find the distance between any two points in a coordinate plane.

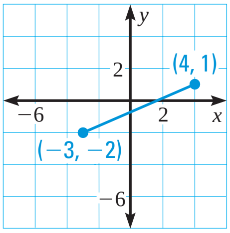
Skill Check ✓

Use the coordinate plane to estimate the distance between the two points. Then use the distance formula to find the distance between the points. Round the result to the nearest hundredth.

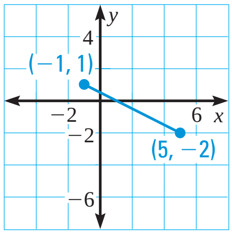
3. (1, 5), (−3, 1)



4. (−3, −2), (4, 1)



5. (5, −2), (−1, 1)



Decide whether the points are vertices of a right triangle.

6. (0, 0), (20, 0), (20, 21)

7. (4, 0), (4, −4), (10, −4)

8. (−2, 0), (−1, 0), (1, 7)

9. (2, 0), (−2, 2), (−3, −5)

Find the midpoint between the two points.

10. (4, 4), (−1, 2)

11. (6, 2), (2, −3)

12. (−5, 3), (−3, −3)

13. **SOCCER** Suppose the soccer ball in Example 3 lands in a position that is 25 yards from the same goal line and 25 yards from the same sideline. How far was the ball kicked?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice
to help you master skills is on p. 808

FINDING DISTANCE **Find the distance between the two points. Round the result to the nearest hundredth if necessary.**

14. (2, 0), (8, −3)

15. (2, −8), (−3, 3)

16. (3, −1), (0, 3)

17. (5, 8), (−2, 3)

18. (−3, 1), (2, 6)

19. (−6, −2), (−3, −5)

20. (4, 5), (−1, 3)

21. (−6, 1), (3, 1)

22. (−2, −1), (3, −3)

23. (3.5, 6), (−3.5, −2)

24. $\left(\frac{1}{2}, \frac{1}{4}\right)$, (2, 1)

25. $\left(\frac{1}{3}, \frac{1}{6}\right), \left(-\frac{2}{3}, \frac{8}{3}\right)$

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 14–25
Example 2: Exs. 26–33
Example 3: Exs. 46–53
Example 4: Exs. 34–45
Example 5: Exs. 54, 55

RIGHT TRIANGLES **Graph the points. Decide whether they are vertices of a right triangle.**

26. (4, 0), (2, 1), (−1, −5)

27. (5, 4), (2, 1), (−3, 2)

28. (1, −5), (2, 3), (−3, 4)

29. (−1, 1), (−3, 3), (−7, −1)

30. (−3, 2), (−3, 5), (0, 2)

31. (3, −1), (2, 4), (−3, 0)

32. (−2, 2), (3, 4), (4, 2)

33. (0, −4), (4, −1), (4, −4)

FINDING THE MIDPOINT **Find the midpoint between the two points.**

34. (3, 0), (−5, 4) **35.** (0, 0), (0, 8) **36.** (1, 2), (5, 4)

37. (−1, 2), (7, 4) **38.** (−3, 3), (2, −2) **39.** (2, 7), (4, 3)

40. (−1, 1), (−4, −4) **41.** (−4, 0), (−1, −5) **42.** (5, 1), (1, −5)

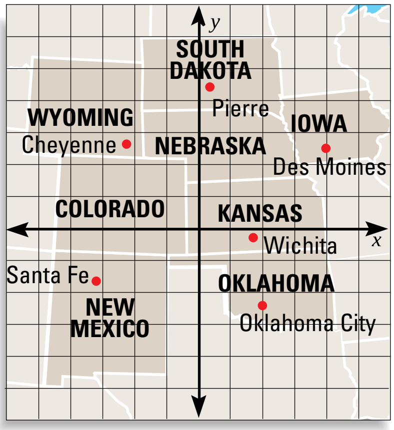
43. (0, −3), (−4, 2) **44.** (5, −5), (−5, 1) **45.** (−4, −3), (−1, −5)

FOCUS ON CAREERS

REAL LIFE **CARTOGRAPHER** Cartographers prepare maps using information from surveys, aerial photographs, and satellite data.

INTERNET **CAREER LINK** www.mcdougallittell.com

DISTANCES ON MAPS **In Exercises 46–48, use the map. Each side of a square in the coordinate plane that is superimposed on the map represents 95 miles. The points represent city locations.**

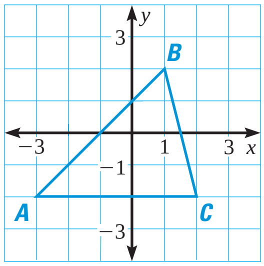


46. Use the distance formula to estimate the distance between Pierre, South Dakota, and Santa Fe, New Mexico.

47. Use the distance formula to estimate the distance between Wichita, Kansas, and Cheyenne, Wyoming.

48. Use the distance formula to estimate the distance between Oklahoma City, Oklahoma, and Des Moines, Iowa.

GEOMETRY CONNECTION **In Exercises 49–53, use the diagram below.**



49. Copy the diagram of triangle *ABC* on graph paper.

50. Find the length of each side of the triangle.

51. Find the midpoint of each side of the triangle.

52. Join the midpoints to form a new triangle. Find the length of each of its sides.

53. Compare the perimeters of the two triangles.

HIKING TRIP **In Exercises 54 and 55, use the following information. You and a friend go hiking. You hike 3 miles north and 2 miles west. Starting from the same point, your friend hikes 4 miles east and 1 mile south.**

54. How far apart are you and your friend? (*Hint*: Draw a diagram on a grid.)

55. If you and your friend want to meet for lunch, where could you meet so that both of you hike the same distance? How far do you have to hike?

GEOMETRY CONNECTION **In Exercises 56–58, use the following information. A trapezoid is *isosceles* if its two opposite nonparallel sides have the same length.**

56. Draw the polygon whose vertices are $A(1, 1)$, $B(5, 9)$, $C(2, 8)$, and $D(0, 4)$.

57. Show that the polygon is a trapezoid by showing that only two of the sides are parallel.

58. Use the distance formula to show that the trapezoid is isosceles.

Test Preparation

59. **MULTIPLE CHOICE** What is the distance between $(-6, -2)$ and $(2, 4)$?

Ⓐ $2\sqrt{5}$ Ⓑ $2\sqrt{7}$ Ⓒ 10 Ⓓ 28

60. **MULTIPLE CHOICE** What is the midpoint between $(-2, -3)$ and $\left(1, \frac{1}{2}\right)$?

Ⓐ $\left(-1, -2\frac{1}{2}\right)$ Ⓑ $\left(-\frac{1}{2}, -2\frac{1}{2}\right)$ Ⓒ $\left(-1, -1\frac{1}{4}\right)$ Ⓓ $\left(-\frac{1}{2}, -1\frac{1}{4}\right)$

61. **MULTIPLE CHOICE** The vertices of a right triangle are $(0, 0)$, $(0, 6)$, and $(6, 0)$. What is the length of the hypotenuse?

Ⓐ 6 Ⓑ $6\sqrt{2}$ Ⓒ 36 Ⓓ 72

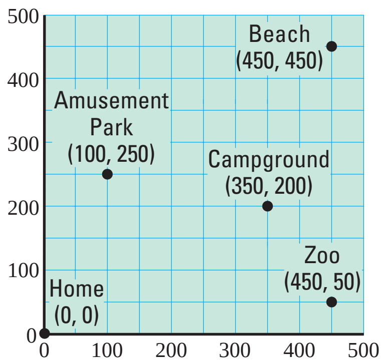
★ Challenge

TRIP PLANNING In Exercises 62–64, use the following information. You are planning a family vacation. Each side of a square in the coordinate plane that is superimposed on the map represents 50 miles.

62. How far is it from your home to the amusement park?

63. You leave your home and go to the amusement park. After visiting the amusement park, you go to the beach. You return home. How far did you travel?

64. During your vacation, you want to visit all of the sites on the map. There are two orders in which to visit the sites so that you travel the shortest distance. What are the two orders?



EXTRA CHALLENGE
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MIXED REVIEW

FACTORING Factor the expression completely. (Review 10.8)

65. $3x^3 + 12x^2 - 15x$

66. $x^4 - 3x^2 - 25x^2 + 75$

CLASSIFYING EQUATIONS Does the equation model *direct variation*, *inverse variation*, or *neither*? (Review 11.3)

67. $x = \frac{7}{y}$

68. $y = 8x$

69. $y = 9x + 1$

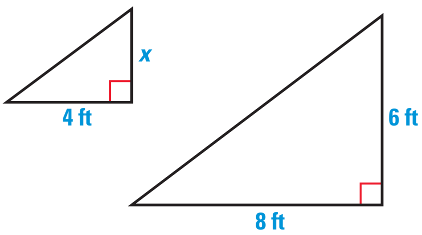
DIVIDING POLYNOMIALS Divide. (Review 11.7)

70. $(6z + 10) \div 2$

71. $(7x^3 - 2x^2) \div 14x$

GEOMETRY CONNECTION **In Exercises 72 and 73, the two triangles are similar. Write an equation and solve it to find the length of the side marked x.** (Review 3.2 for 12.7)

72.



73.